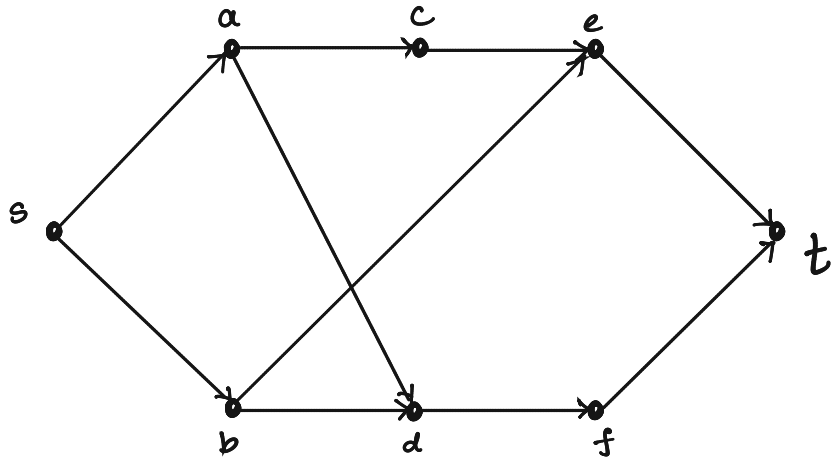
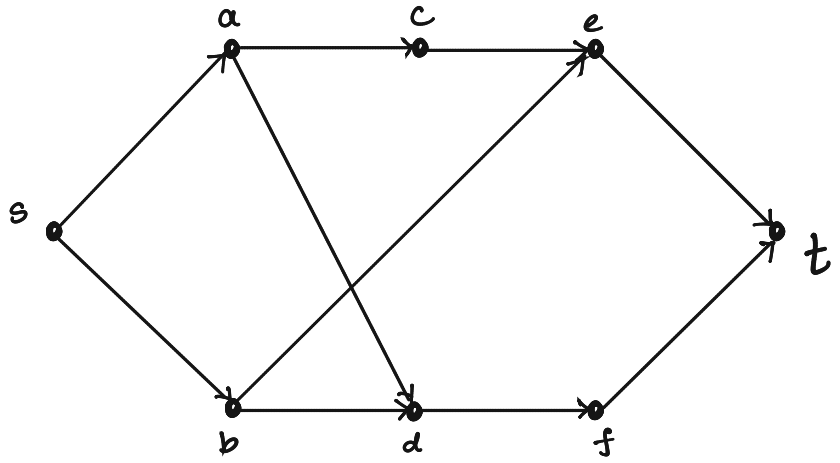


A DIRECTED GRAPH

s ← SOURCE

t ← SINK

$\{a, b, c, d, e, f\}$ ← INTERNAL NODES

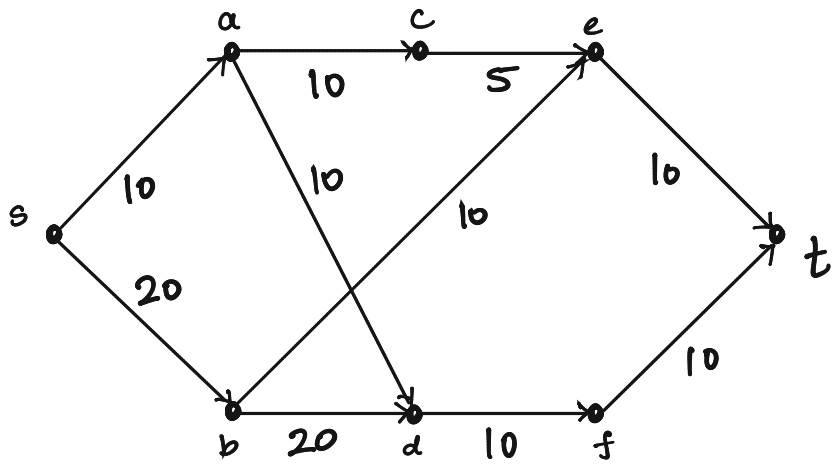


A DIRECTED GRAPH

s ← SOURCE [NO INCOMING EDGE]

t ← SINK [NO OUTGOING EDGE]

{a, b, c, d, e, f} ← INTERNAL NODES



A DIRECTED GRAPH

s ← SOURCE [NO INCOMING EDGE]

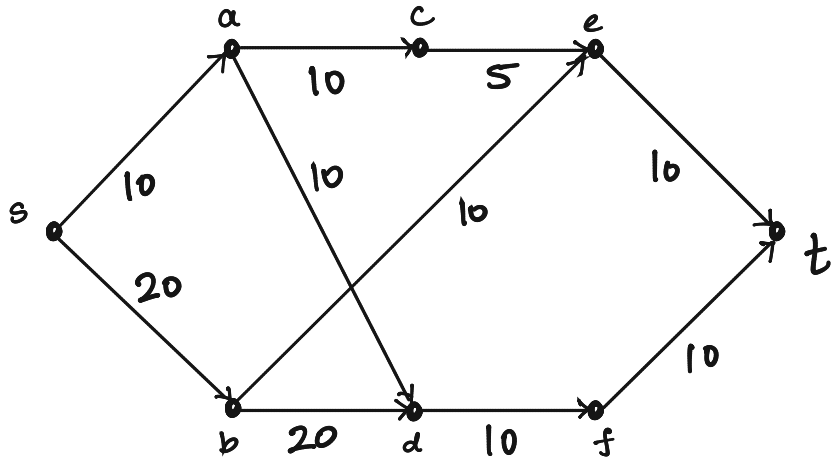
t ← SINK [NO OUTGOING EDGE]

$\{a, b, c, d, e, f\}$ ← INTERNAL NODES

EACH EDGE HAS SOME CAPACITY

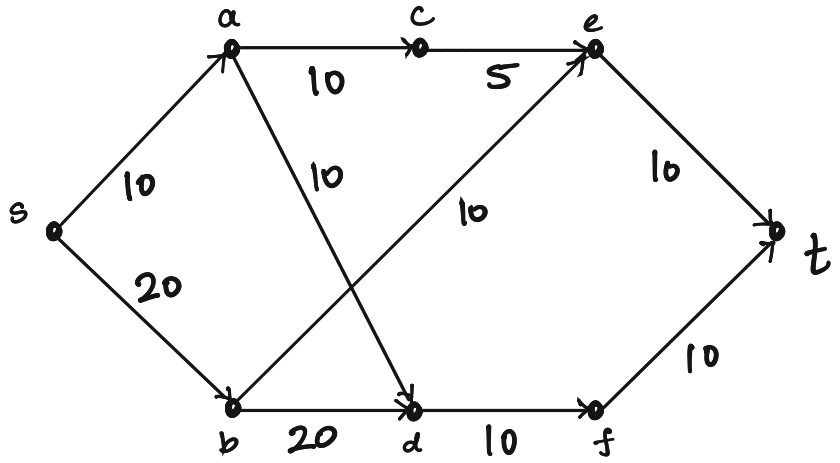
$c : E \to \mathbb{R}^+$

$c(e)$ ← CAPACITY OF EDGE e



→ s WANTS TO SEND SOME OBJECTS TO t.

→ # OBJECT SENT ON AN EDGE CANNOT EXCEED ITS CAPACITY.

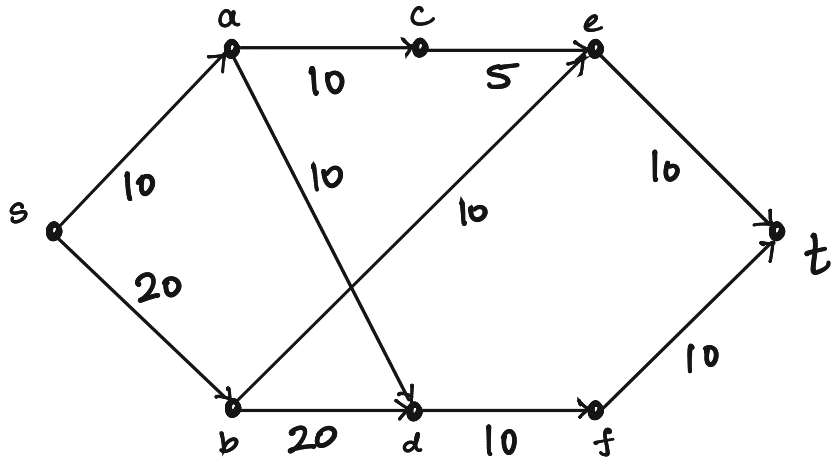


→ s WANTS TO SEND SOME OBJECTS TO t.

→ # OBJECT SENT ON AN EDGE CANNOT EXCEED ITS CAPACITY

OBJECTS SENT THROUGH e

$\leq c(e)$



→ s WANTS TO SEND SOME OBJECTS TO t.

→ # OBJECT SENT ON AN EDGE CANNOT EXCEED ITS CAPACITY

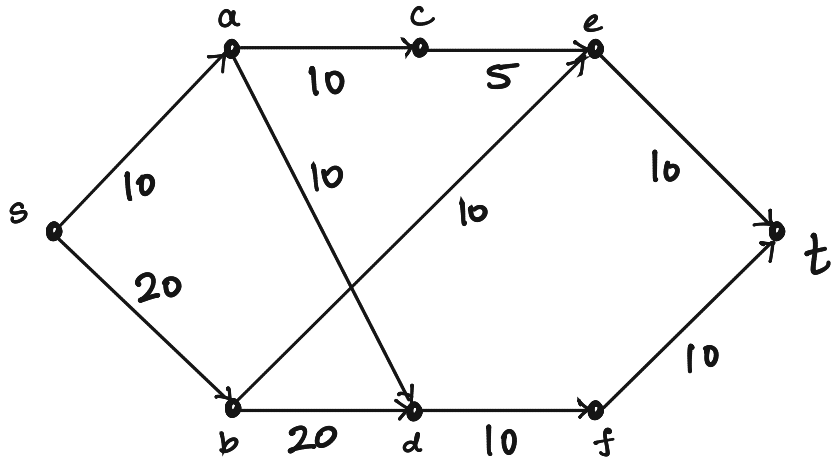
OBJECTS SENT THROUGH e

$\leq$ c(e)

f(e) $\leq$ c(e) (CAPACITY CONSTRAINT)

↑

FLOW THROUGH e.



→ s WANTS TO SEND SOME OBJECTS TO t.

→ # OBJECT SENT ON AN EDGE CANNOT EXCEED ITS CAPACITY

OBJECTS SENT THROUGH e

$\leq$ c(e)

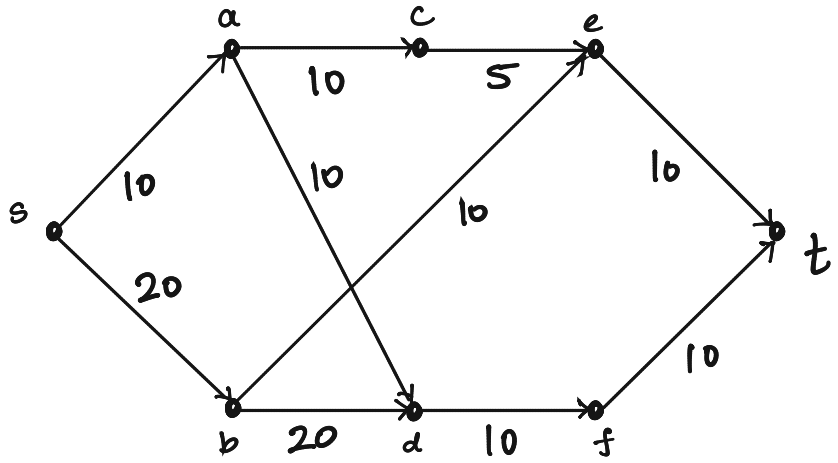
f(e) $\leq$ c(e) (CAPACITY CONSTRAINT)

↑

FLOW THROUGH e.

→ # OBJECTS RECIVED BY v

= # OBJECTS SENT OUT BY v.



→ s WANTS TO SEND SOME OBJECTS TO t.

→ # OBJECT SENT ON AN EDGE CANNOT EXCEED ITS CAPACITY

OBJECTS SENT THROUGH e

$\leq$ c(e)

f(e) $\leq$ c(e) (CAPACITY CONSTRAINT)

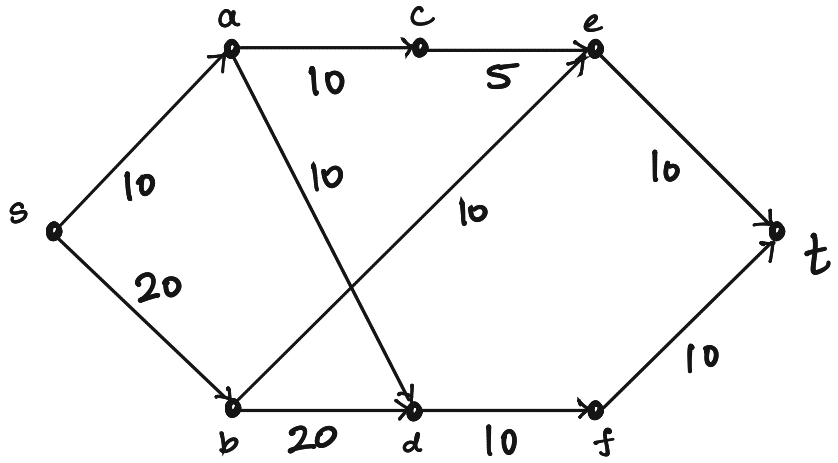
↑

FLOW THROUGH e.

→ # OBJECTS RECIVED BY v

= # OBJECTS SENT OUT BY v.

(TRUE FOR ALL VERTICES EXCEPT s & t)



→ s WANTS TO SEND SOME OBJECTS TO t.

→ # OBJECT SENT ON AN EDGE CANNOT EXCEED ITS CAPACITY

OBJECTS SENT THROUGH e

$\leq$ c(e)

$f(e) \leq c(e)$ (CAPACITY CONSTRAINT)

↑

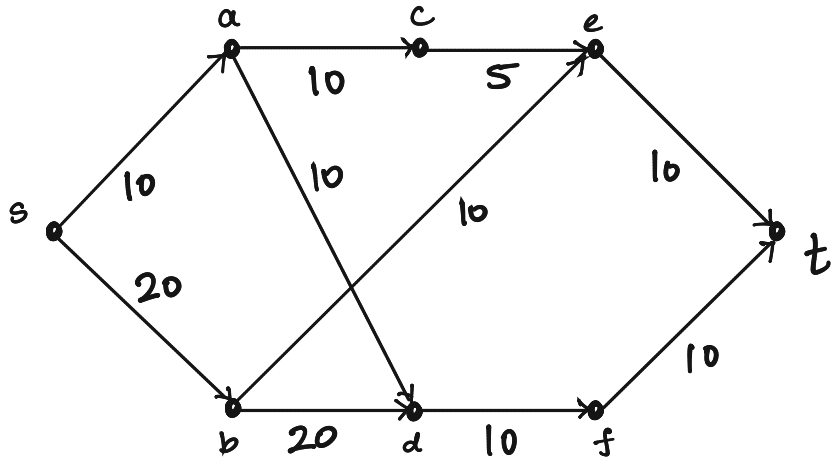
FLOW THROUGH e.

→ # OBJECTS RECIVED BY v

= # OBJECTS SENT OUT BY v.

(TRUE FOR ALL VERTICES EXCEPT s & t)

$$\sum_{e \text{ INCOMING TO } v} f(e) = \sum_{e \text{ OUTGOING FROM } v} f(e)$$ (CONSERVATION OF FLOW)



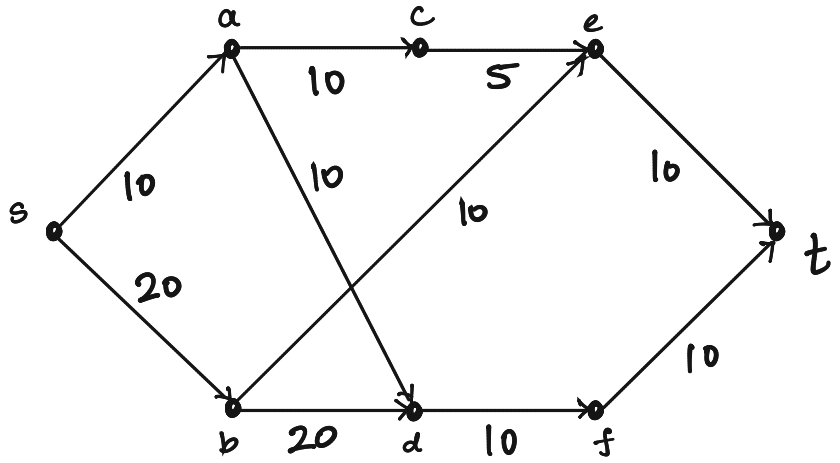
FIND THE MAXIMUM FLOW FROM s to t

SUBJECT TO

(1) $f(e) \leq c(e)$ (CAPACITY CONSTRAINT)

(2) $\sum_{e \text{ INCOMING TO } v} f(e) = \sum_{e \text{ OUTGOING FROM } v} f(e)$

(CONSERVATION OF FLOW)

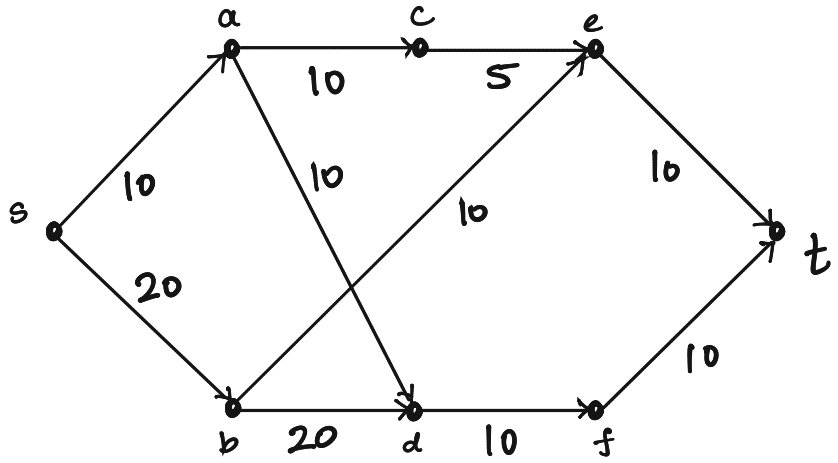


maximize $\sum_{e \text{ OUTGOING FROM } s} f(e)$

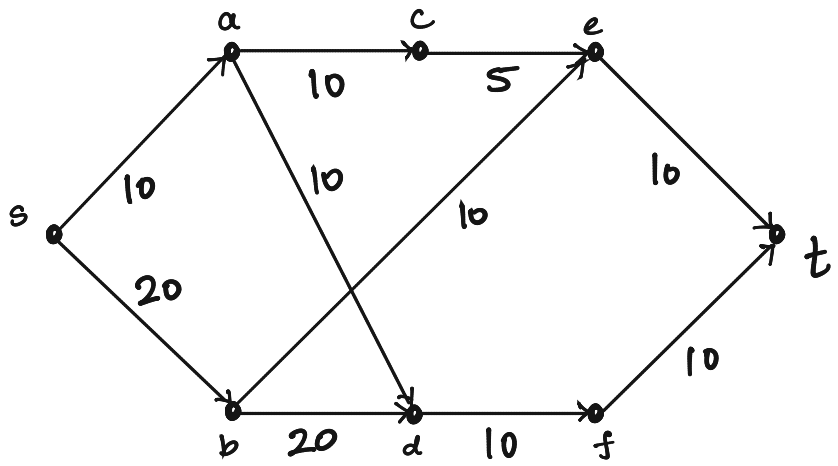
(1) $f(e) \leq c(e)$ (CAPACITY CONSTRAINT)

(2) $\sum_{e \text{ INCOMING TO } v} f(e) = \sum_{e \text{ OUTGOING FROM } v} f(e)$

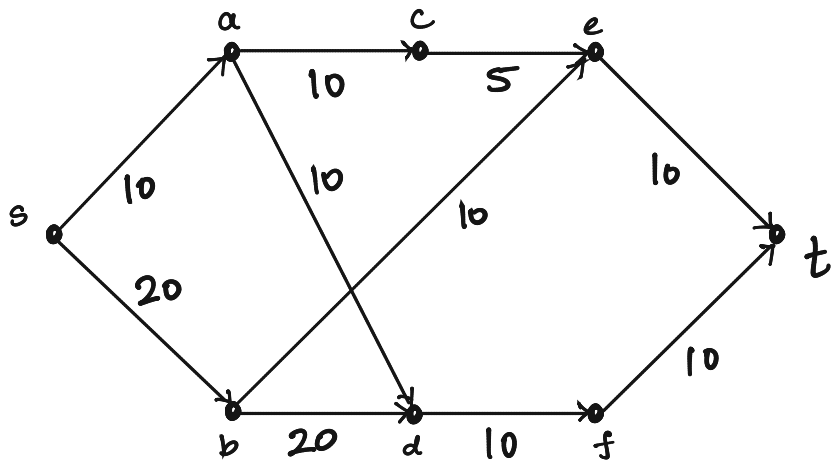
(CONSERVATION OF FLOW)



BE GREEDY :

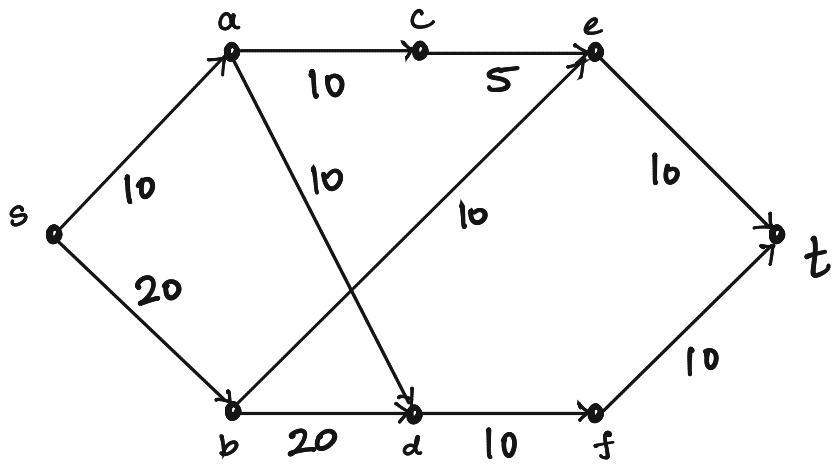


BE GREEDY: "PUSH" AS MUCH FLOW AS YOU CAN ALONG A PATH



BE GREEDY: "PUSH" AS MUCH FLOW AS YOU CAN ALONG A PATH

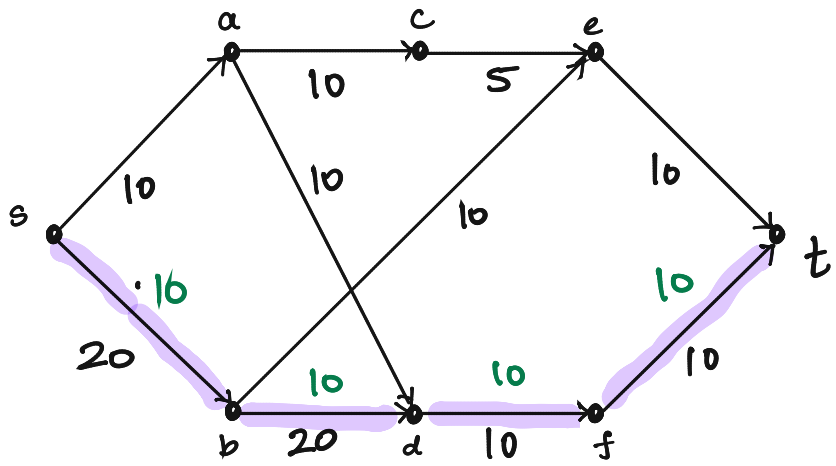
→ FIND A st PATH THAT MAXIMIZES THE MINIMUM EDGE WEIGHT ON THE PATH



BE GREEDY: "PUSH" AS MUCH FLOW AS YOU CAN ALONG A PATH

→ FIND A st PATH THAT MAXIMIZES THE MINIMUM EDGE WEIGHT ON THE PATH

→ MAXIMUM BOTTLENECK PATH FROM s to t.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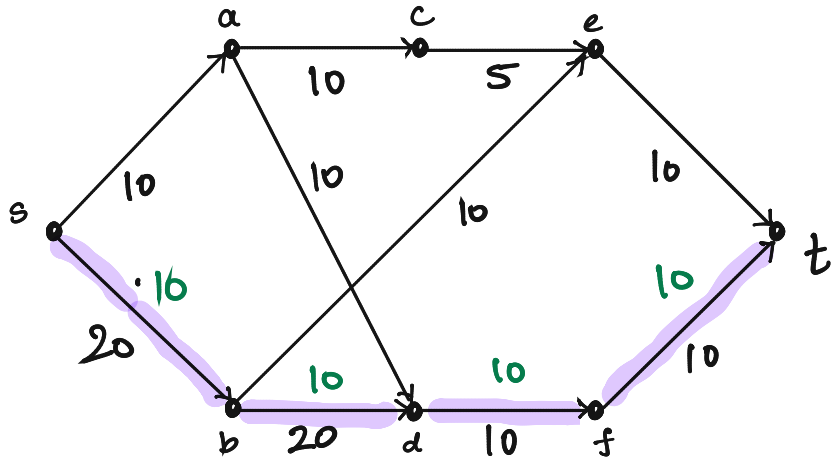
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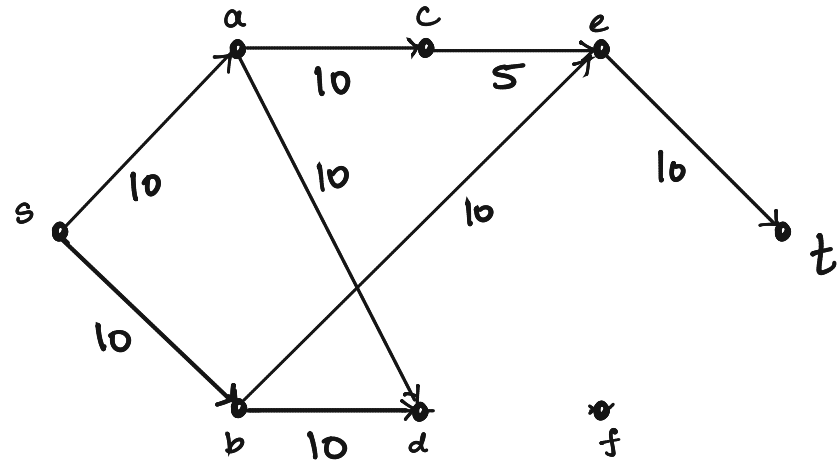
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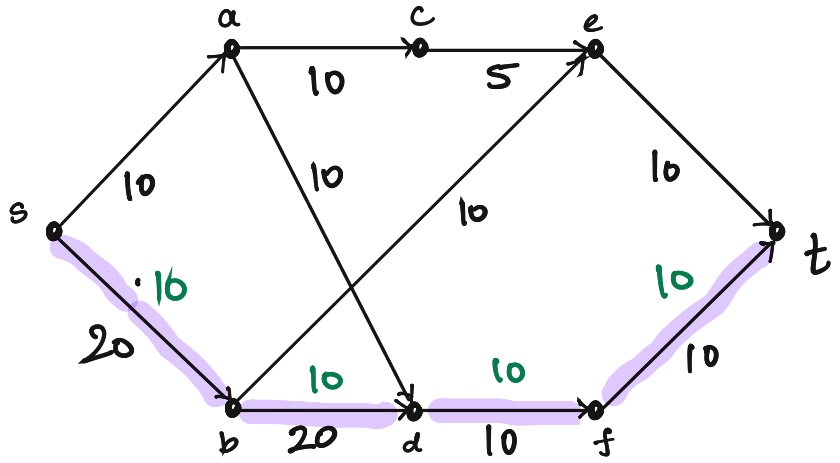
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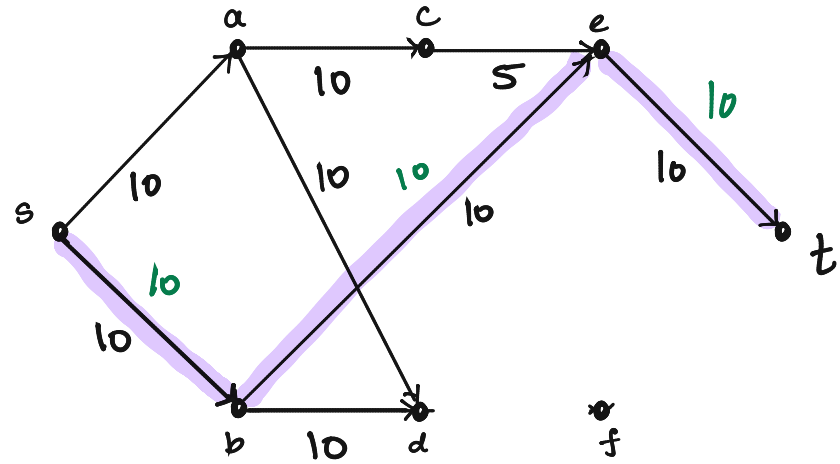
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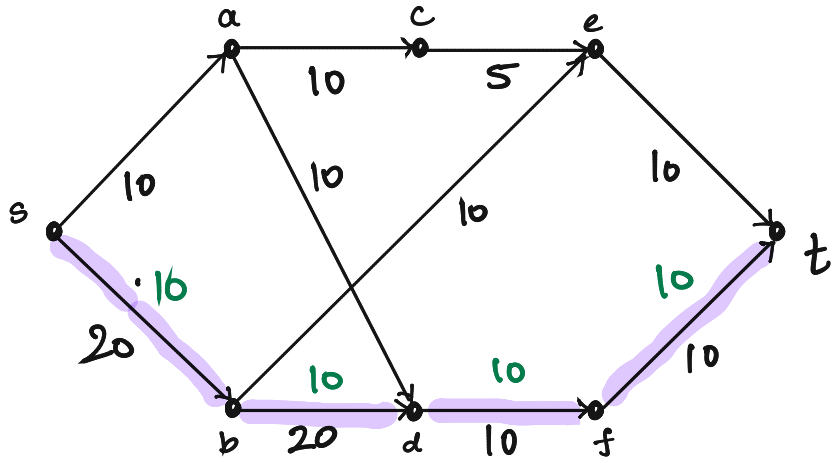
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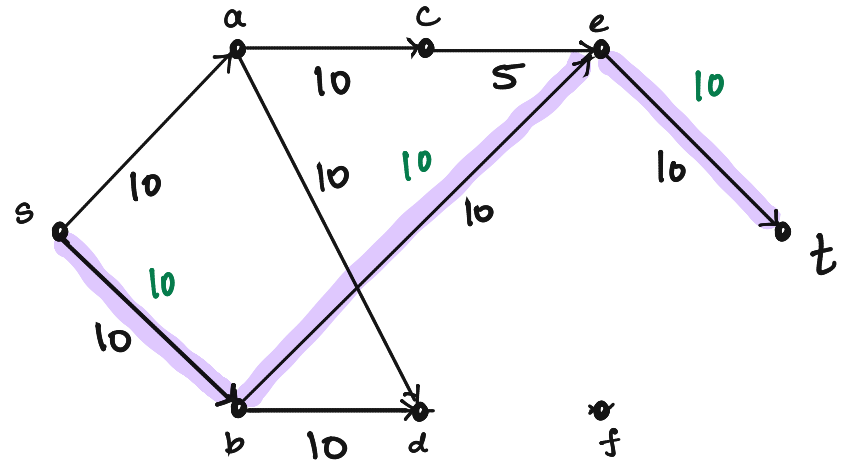
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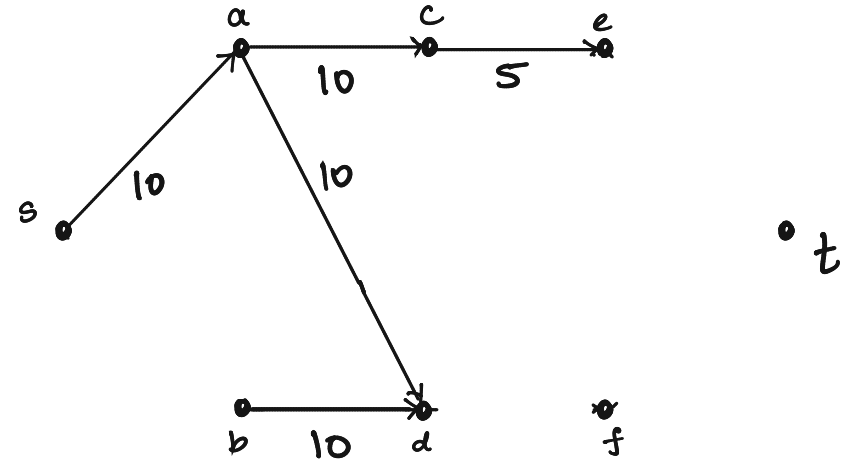
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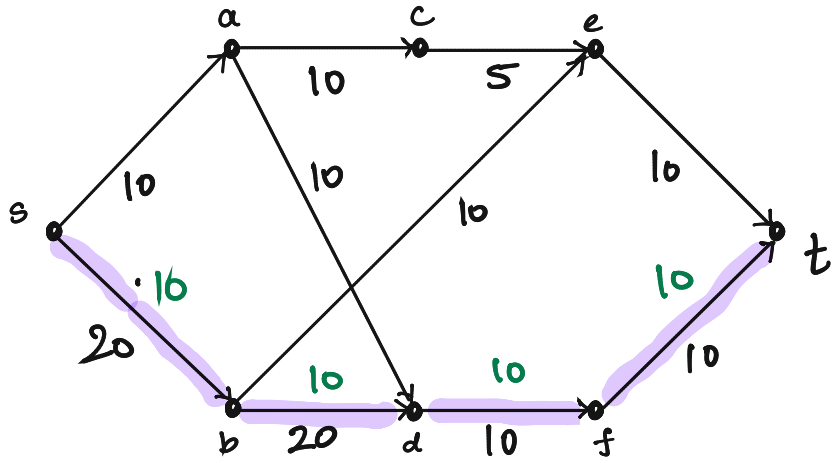
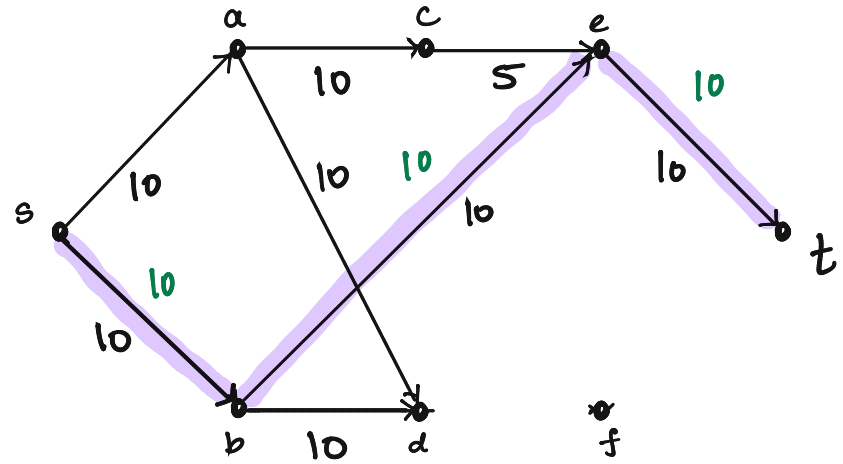
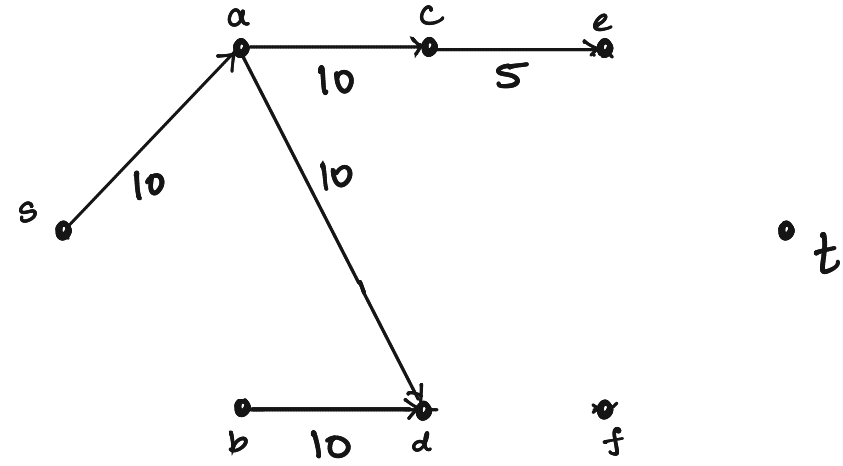
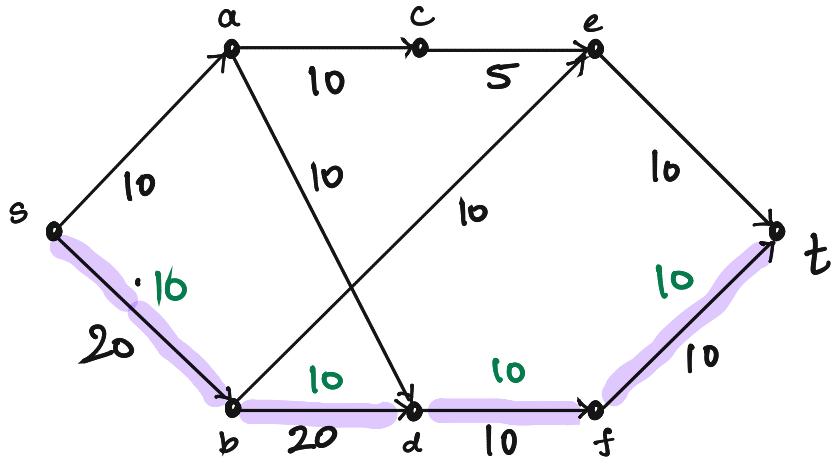


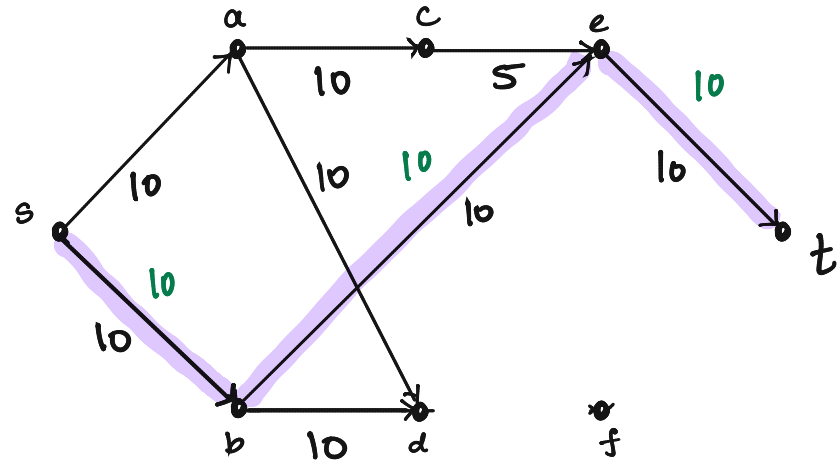


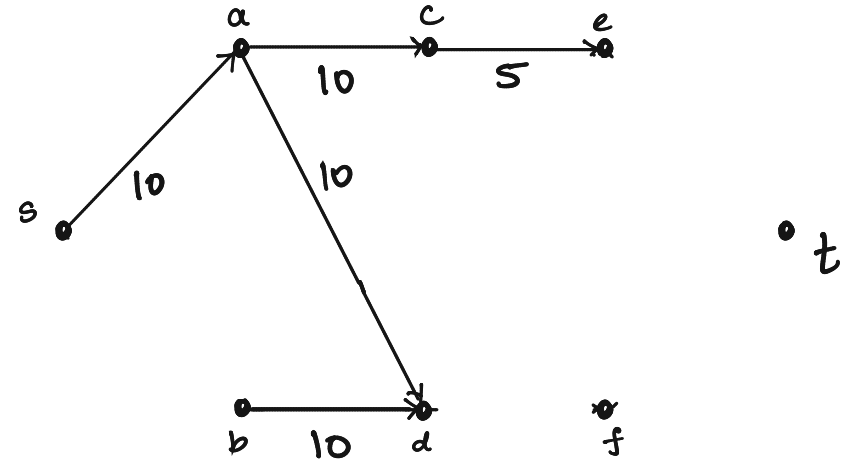


→ NO PATH LEFT FROM s to t

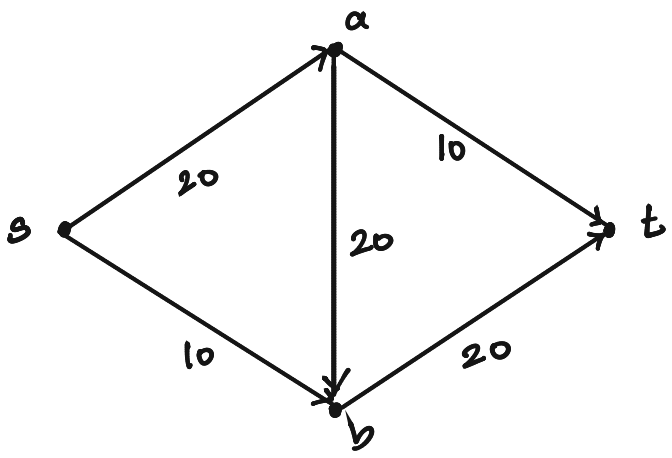
→ PUSHED A FLOW OF 20.







Q: Is this algorithm correct?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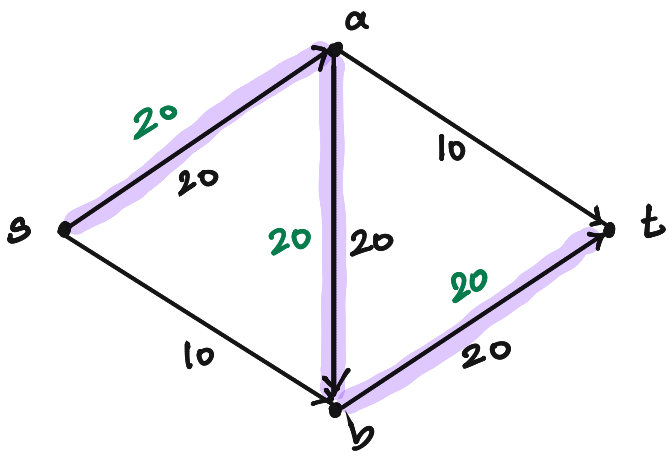
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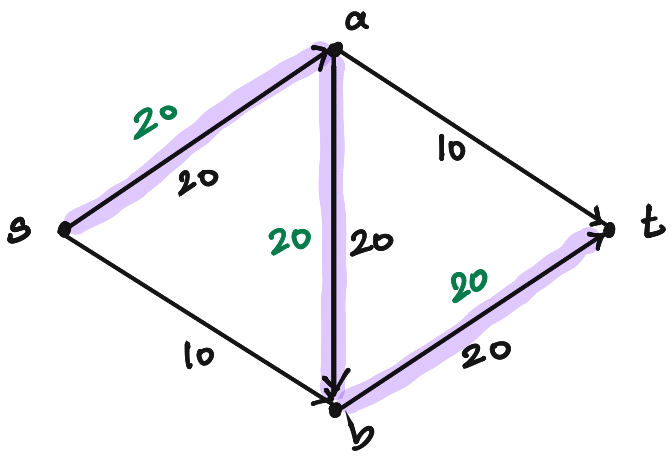
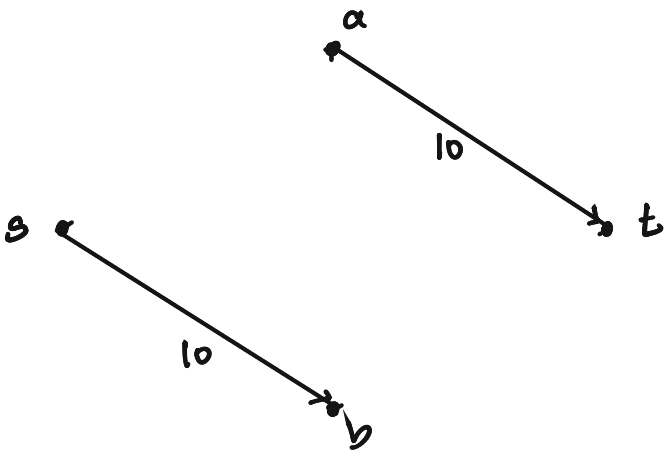
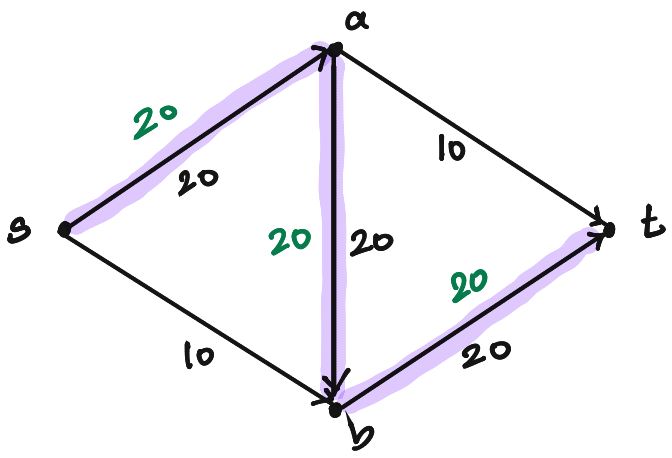
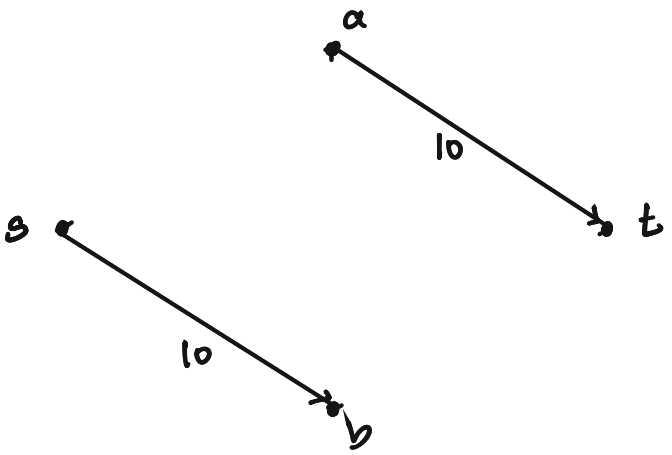




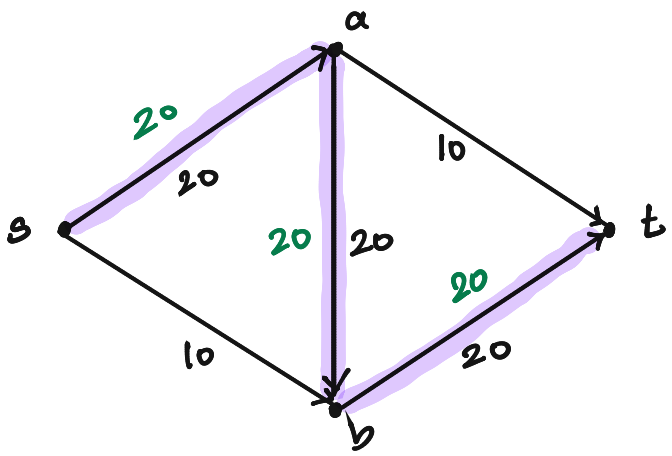
NO MORE PATH FROM s to t.

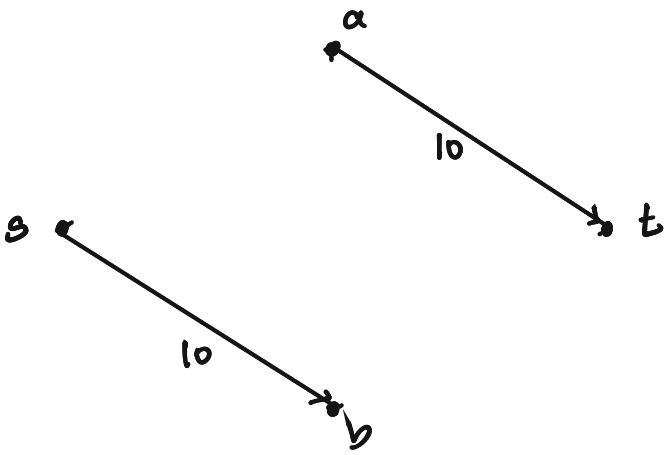




NO MORE PATH FROM s to t.

FLOW OUT OF S : 20.

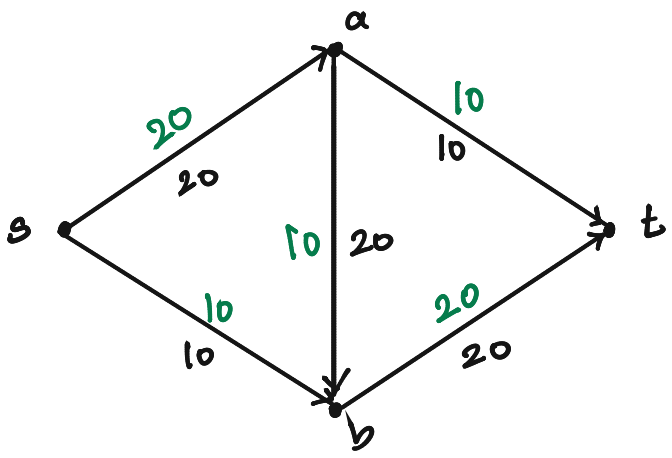




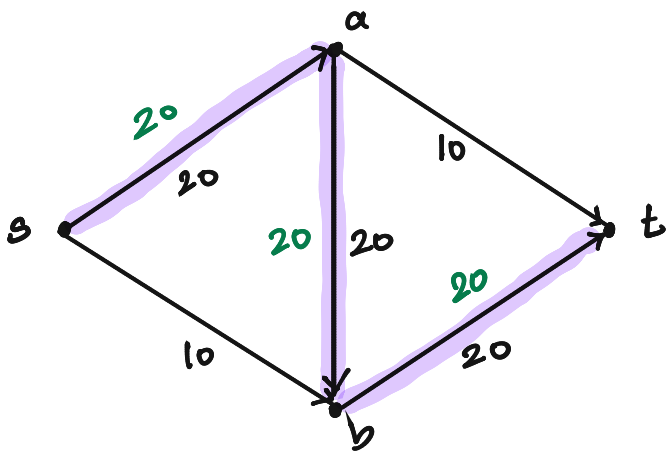
NO MORE PATH FROM s to t.

FLOW OUT OF S : 20.

Q: Is 20 THE MAXIMUM FLOW FROM s to t?



MAXIMUM FLOW = 30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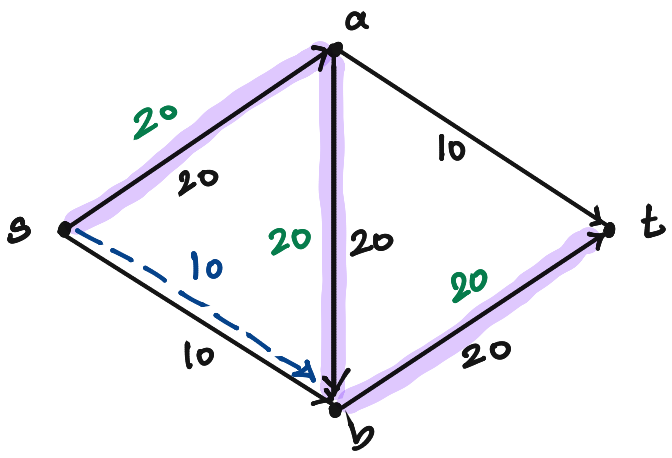
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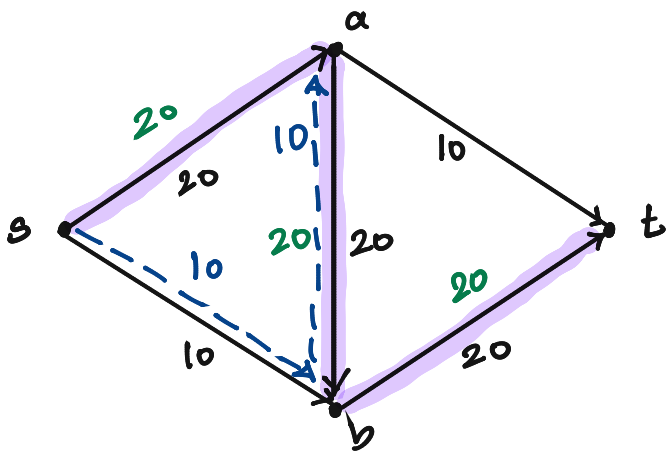
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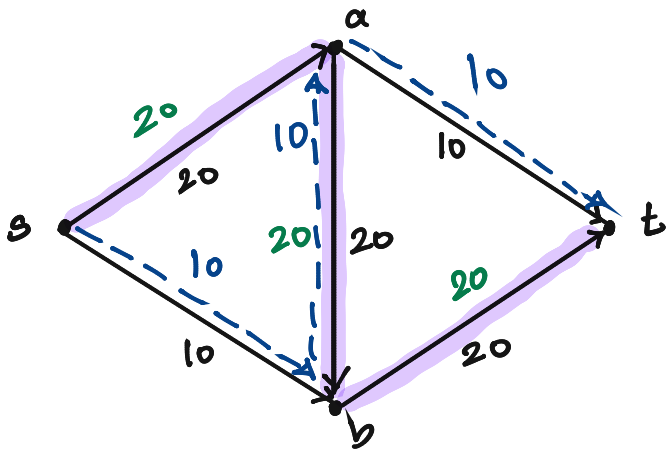
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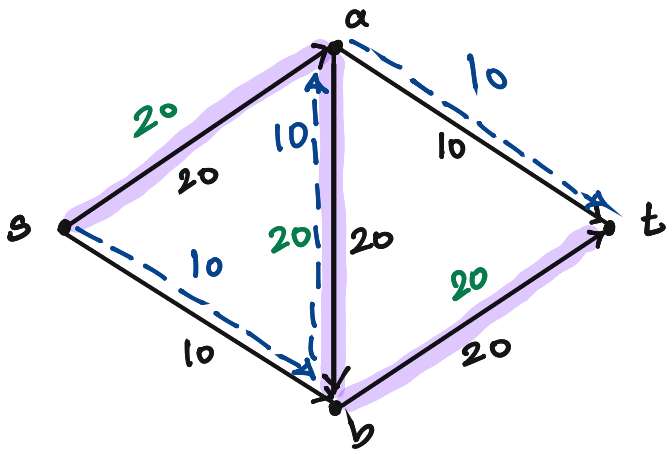
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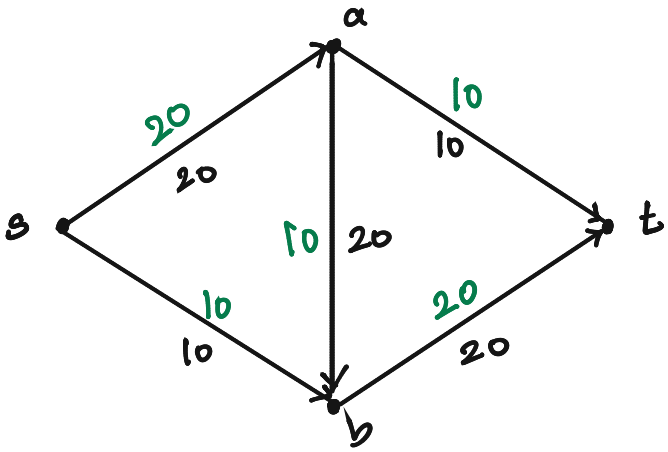
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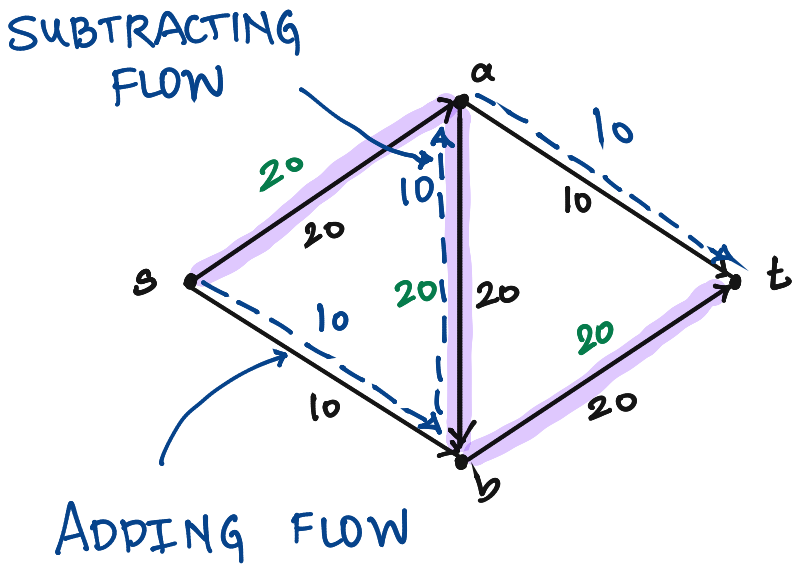
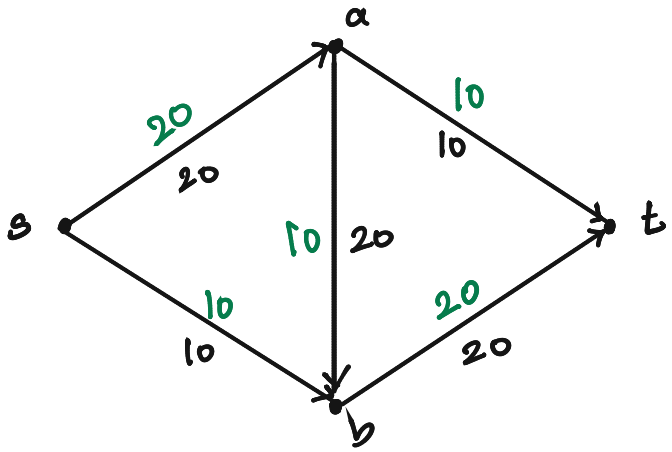
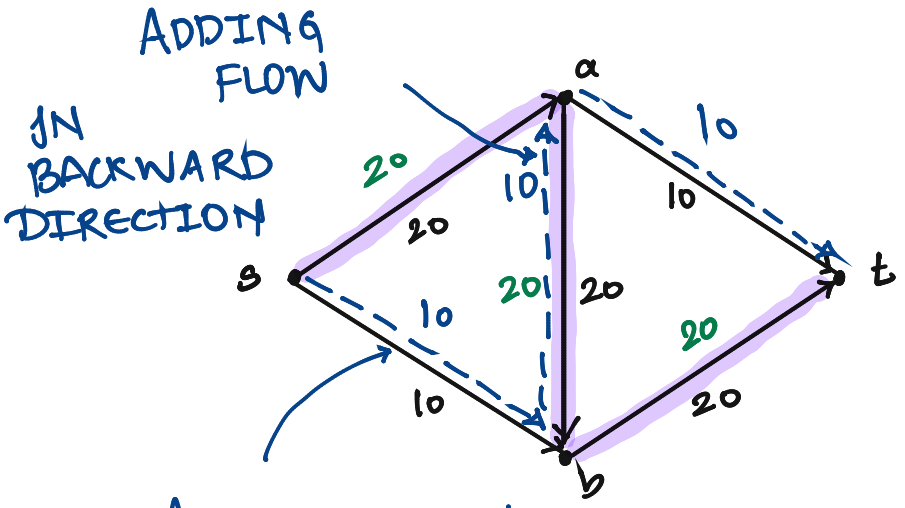


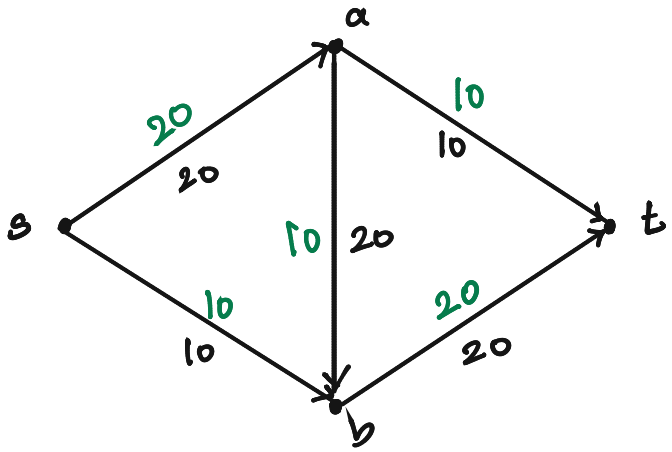


A SLIGHT CHANGE IN GREEDY APPROACH

NEEDS SOME MORE FORMALIZATION.

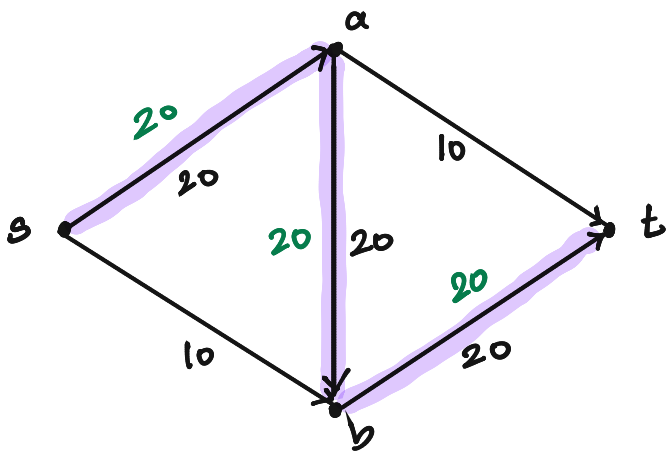


ADDING FLOW IN FORWARD DIRECTION

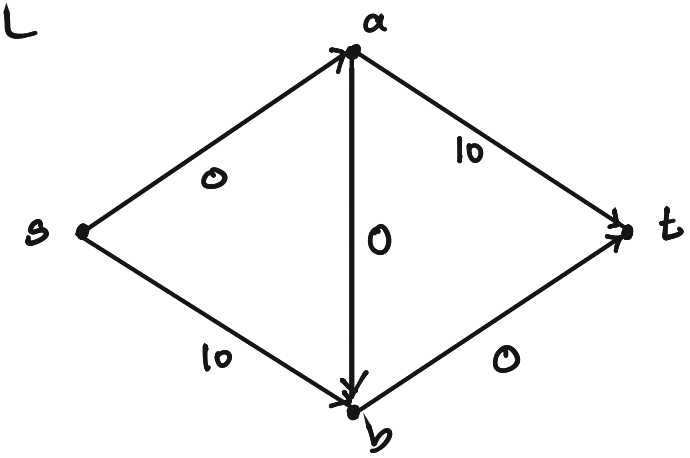


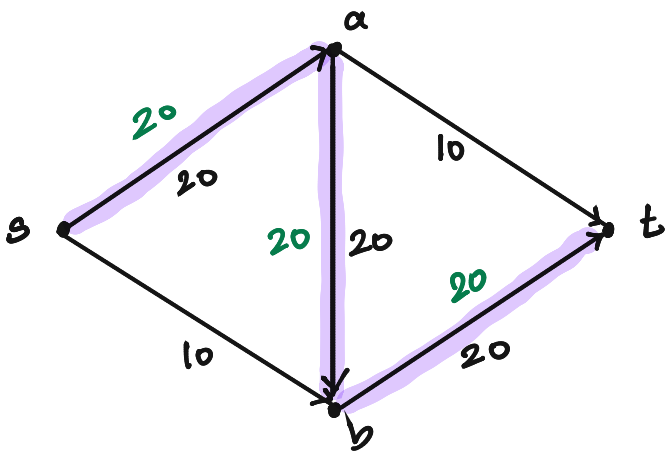
A SLIGHT CHANGE IN GREEDY APPROACH

NEEDS SOME MORE FORMALIZATION.

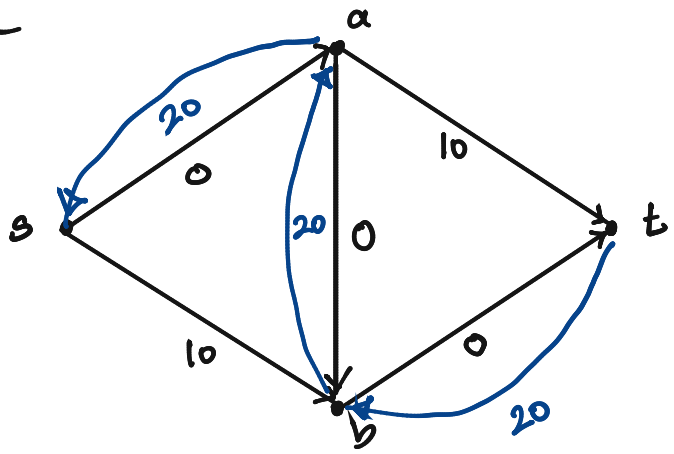


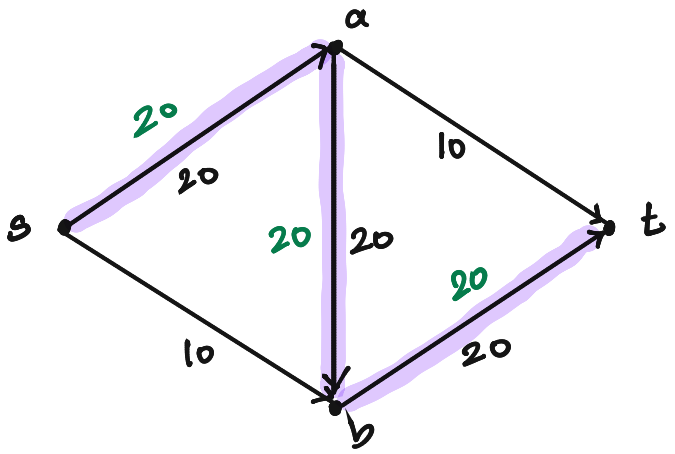
RESIDUAL GRAPH





RESIDUAL GRAPH





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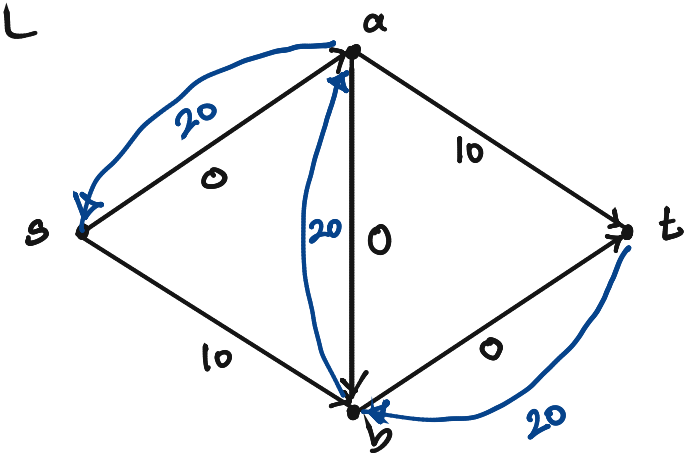
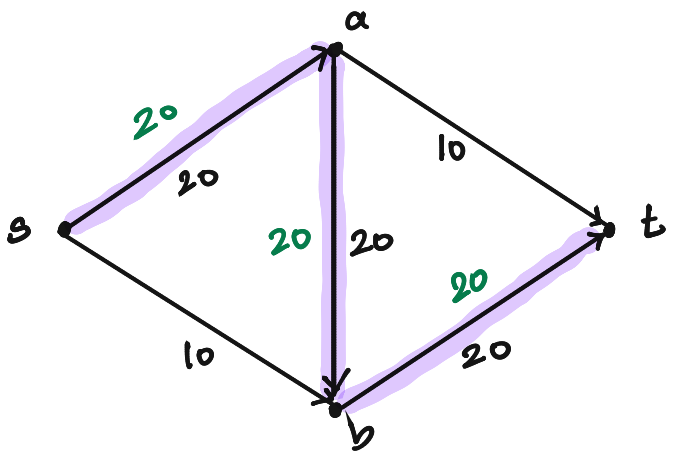
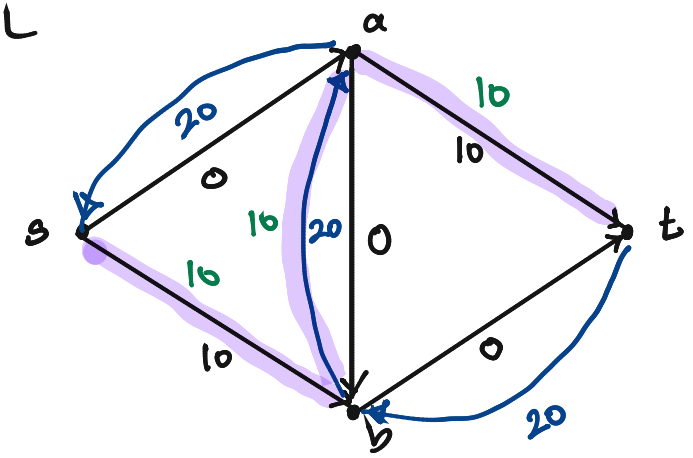


FIND MAXIMUM BOTTLENECK PATH IN THE RESIDUAL GRAPH.



RESIDUAL GRAPH



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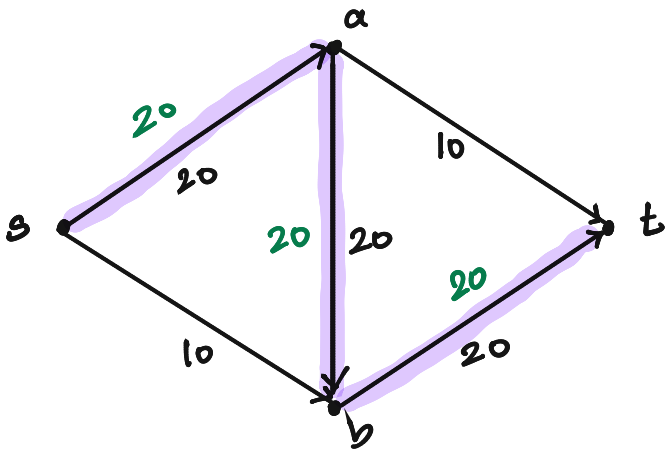

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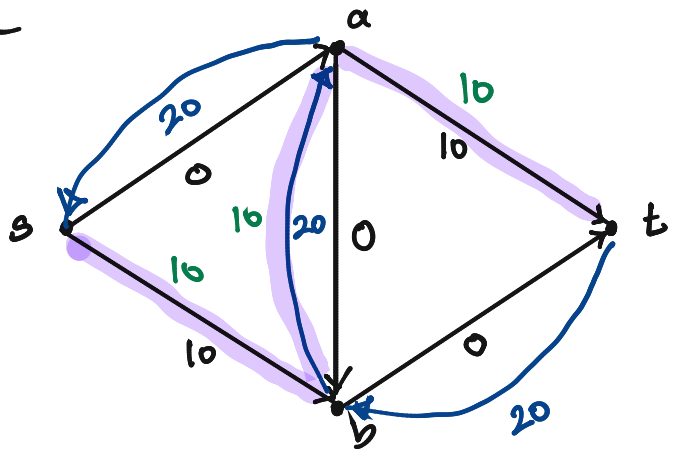

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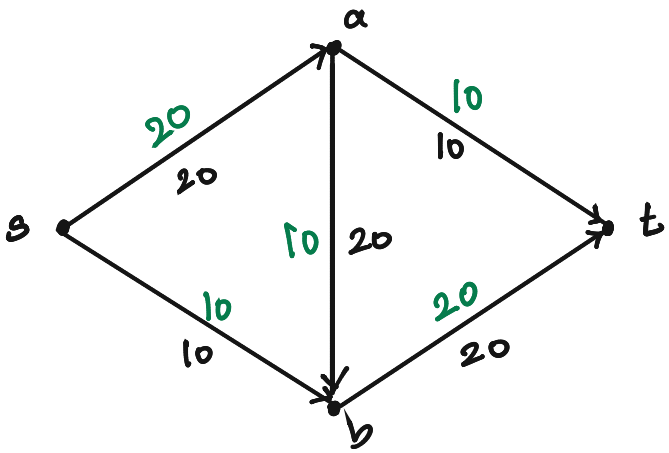

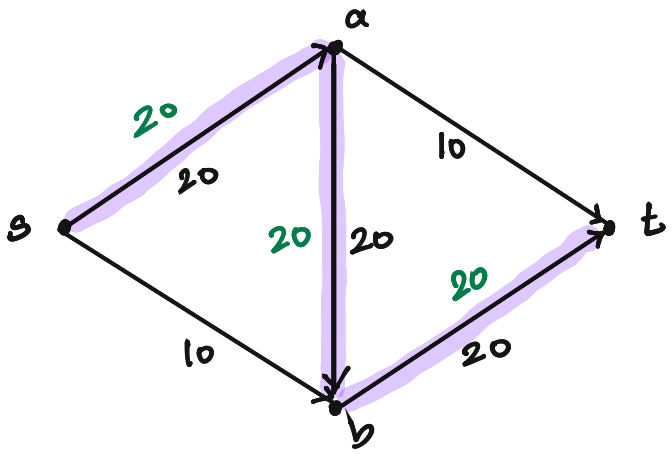

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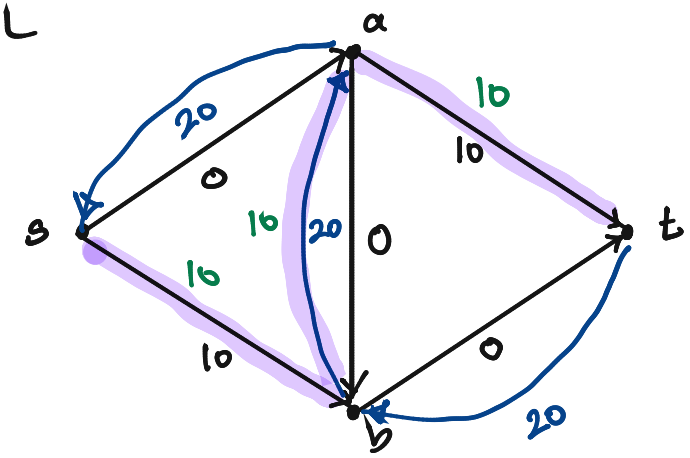

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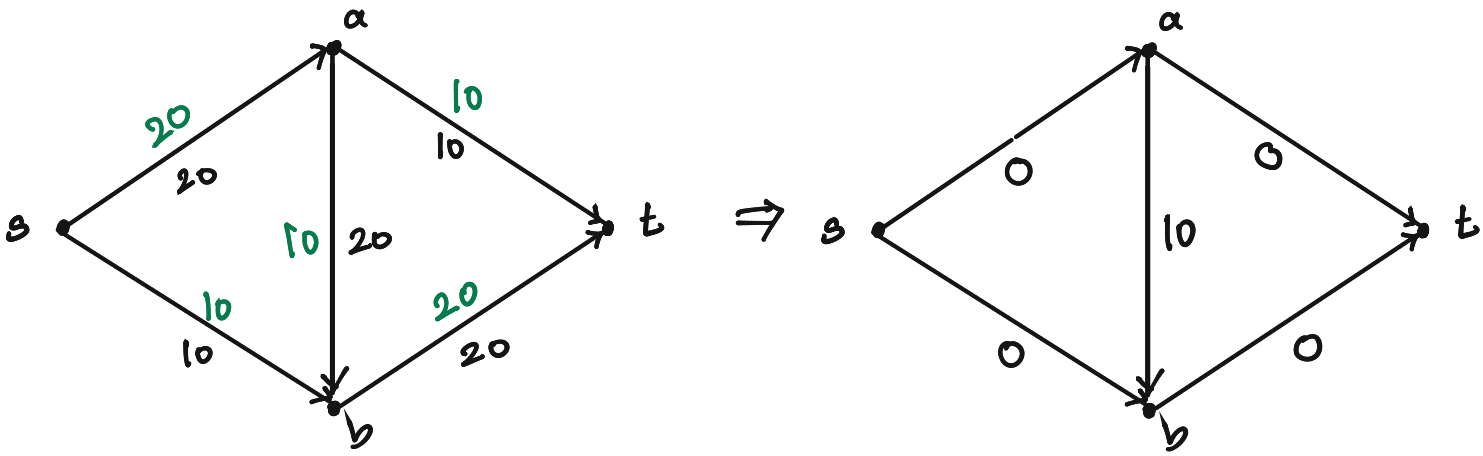

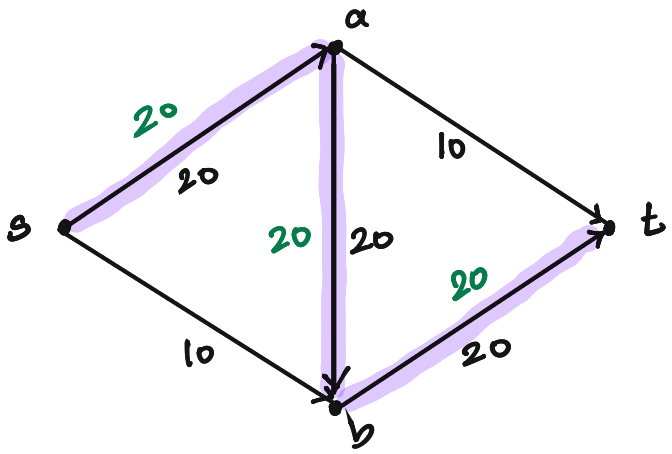

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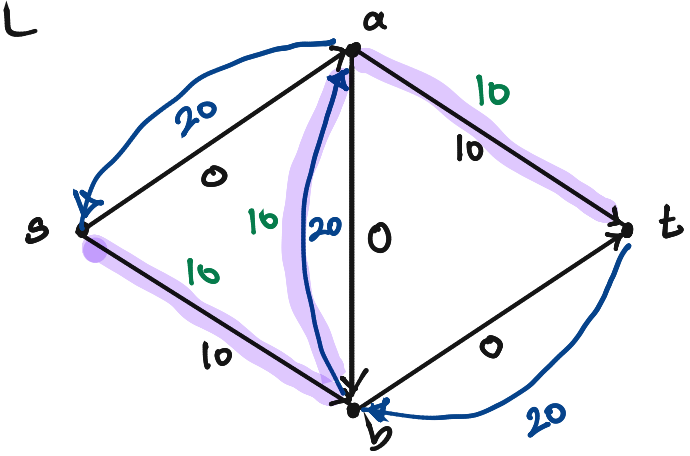

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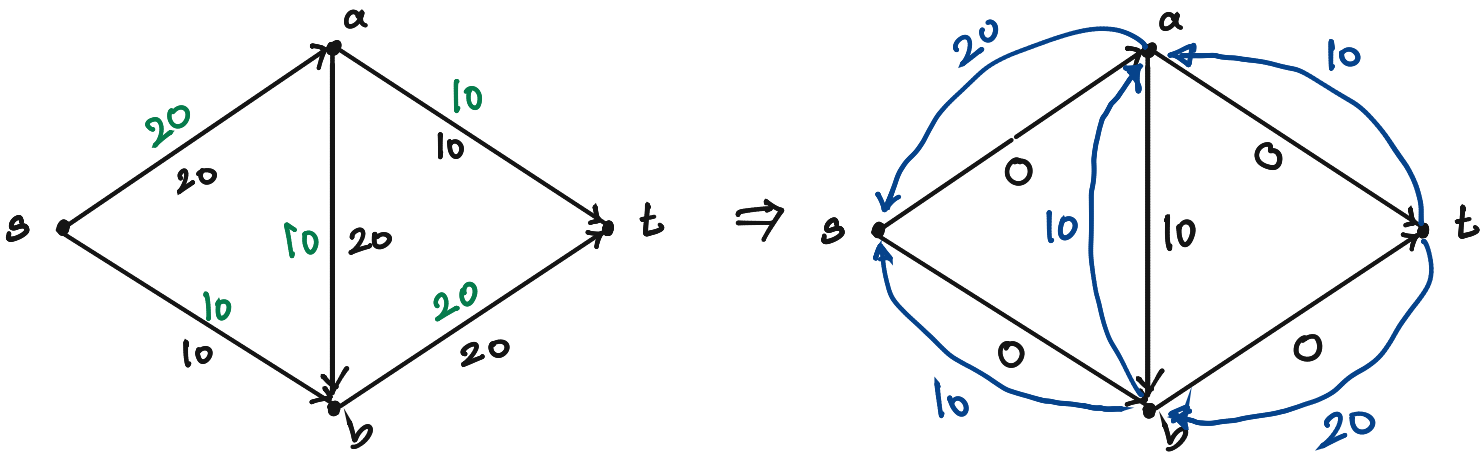

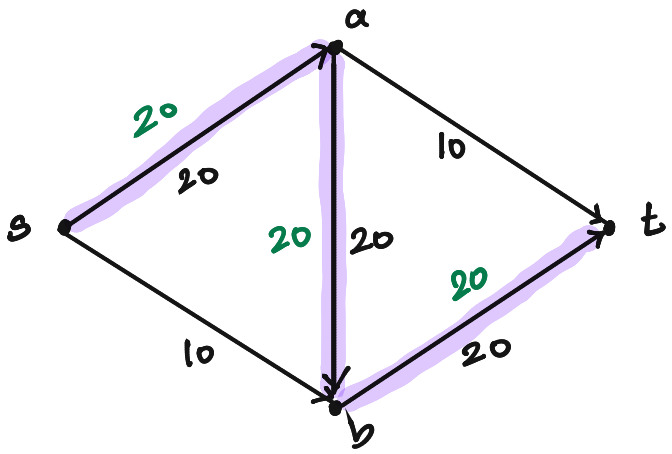

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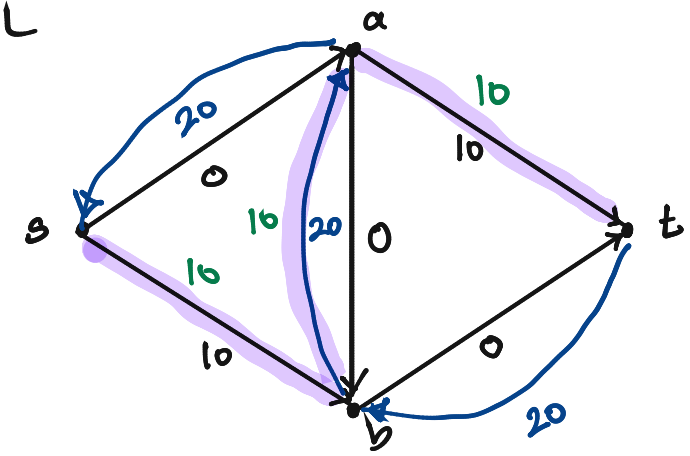

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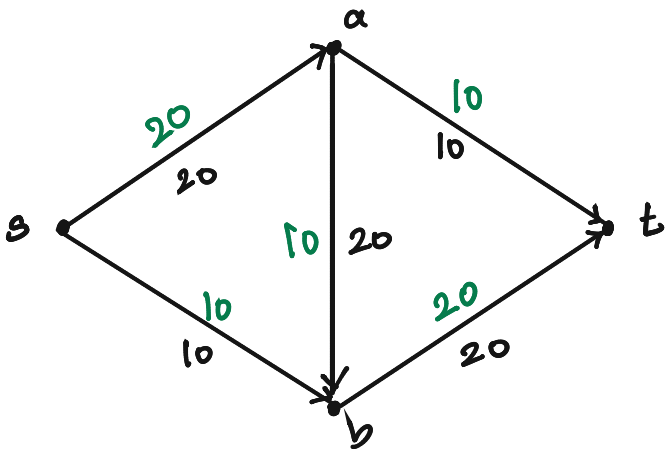

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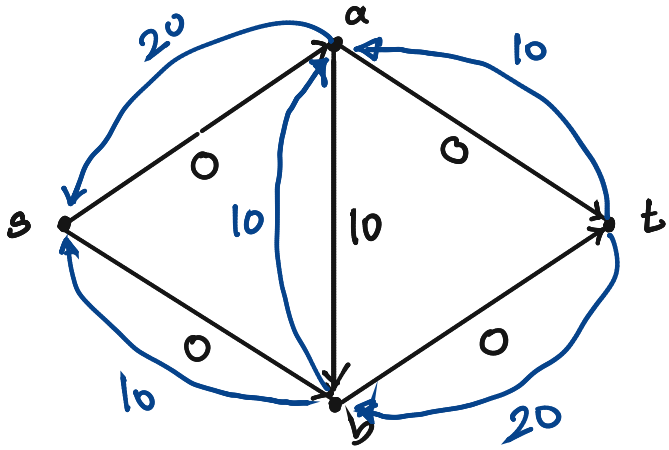

NO st PATH IN RESIDUAL GRAPH

RESIDUAL GRAPH (G_R)

① VERTICES OF G_R = VERTICES OF G.

② FOR AN EDGE e,

$c_R(e) \leftarrow$

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Q: GIVEN A FLOW $f: E \to R^+$ (WHICH MAY NOT BE MAXIMUM), WE CAN FIND G_R IN .....

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```
MAX-FLOW (G)
{   f ← INITIAL FLOW
    GR ← INITIAL RESIDUAL GRAPH.

    WHILE (
```

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```

f

f'

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(1) $f'(e) \leq c(e) \quad \forall e \in E$ (CAPACITY CONSTRAINT)

(2) $\sum_{e \text{ IN } v} f'(e) = \sum_{e \text{ OUT OF } v} f'(e)$

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Q: WHAT IS CAPACITY OF e IN G_R ?

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A: $b \leq c_R(e) = c(e) - f(e)$

$$f'(e) = f(e) + b \leq f(e) + (c(e) - f(e))$$

$$\Rightarrow \quad f'(e) \leq c(e)$$

LEMMA: IF f IS A VALID FLOW, THEN f' IS ALSO A VALID FLOW.

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e: CORRESPONDING FORWARD EDGE

SINCE f IS A VALID FLOW

$f(e) \leq c(e)$

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$f(e) \leq c(e)$

AND

$f'(e) = f(e) - b$

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$\Rightarrow f'(e) \leq c(e)$

Q: IS $f'(e) \geq 0$?

$c_R(e') = f(e)$

$b \leq c_R(e') = f(e)$

$\Rightarrow f'(e) = f(e) - b \geq 0$

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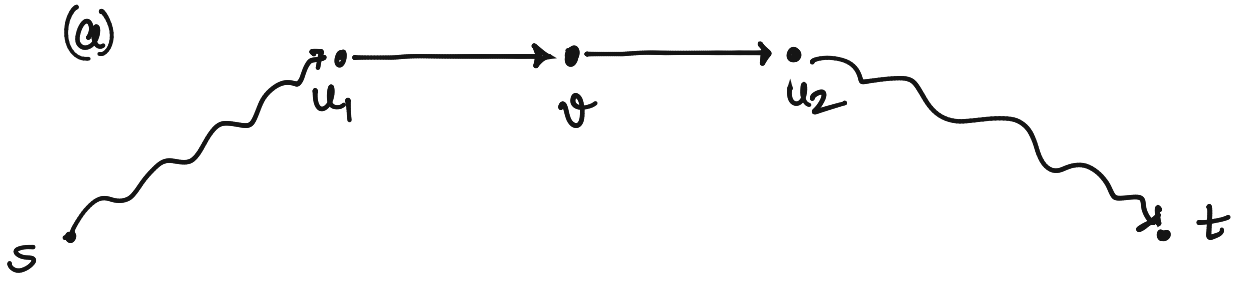
(3) FLOW CONSERVATION

(a)



LEMMA: IF f IS A VALID FLOW, THEN f' IS ALSO A VALID FLOW.

PROOF: TO SHOW

(1) $f'(e) \leq c(e) \quad \forall e \in E$ (CAPACITY CONSTRAINT)

(2) $\sum_{e \text{ IN } v} f'(e) = \sum_{e \text{ OUT OF } v} f'(e)$

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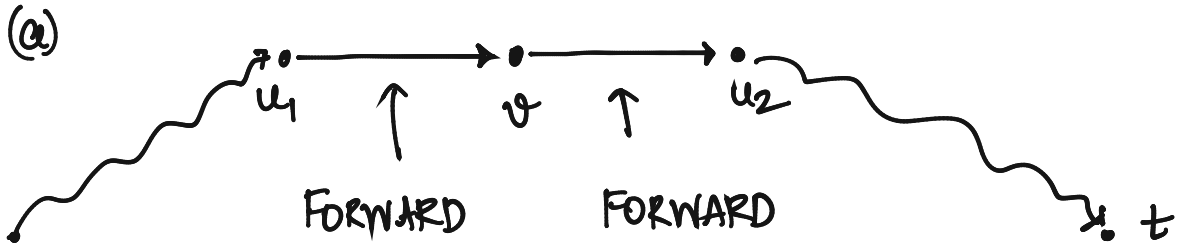

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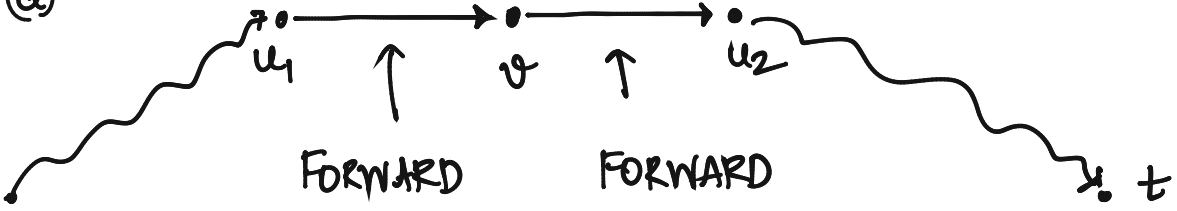

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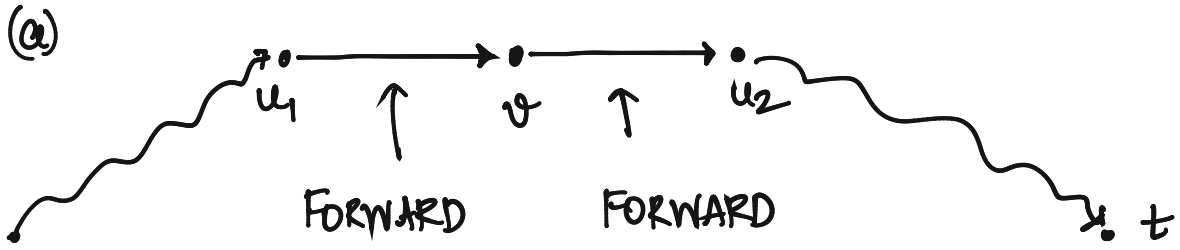

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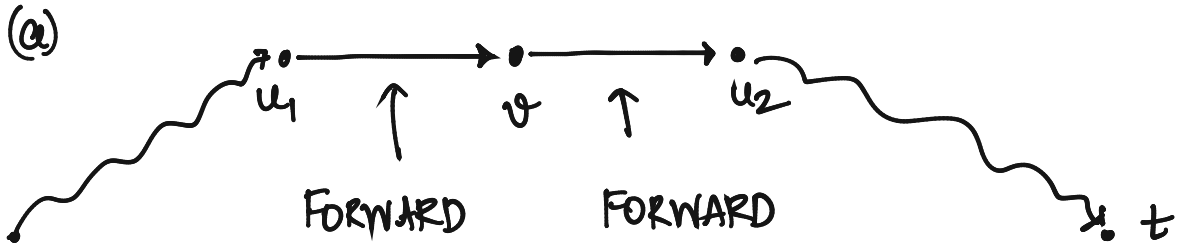

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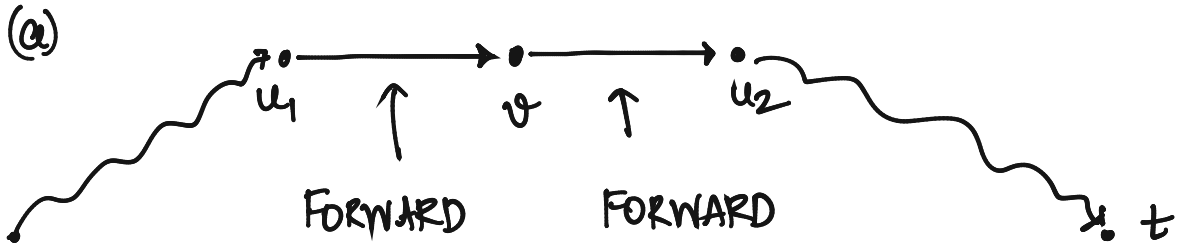

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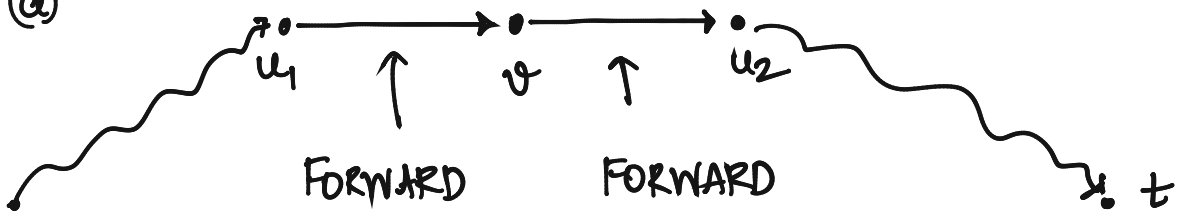

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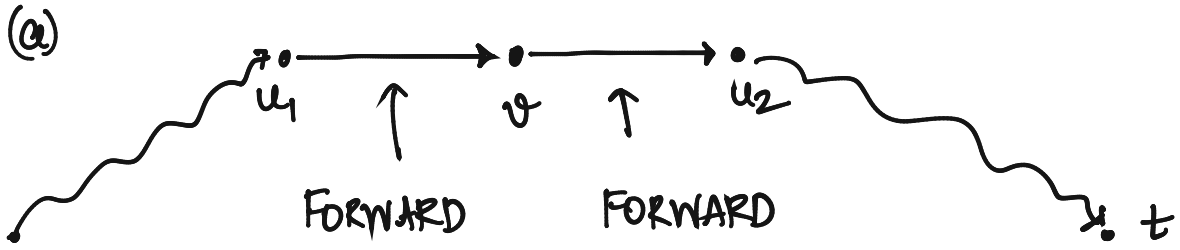

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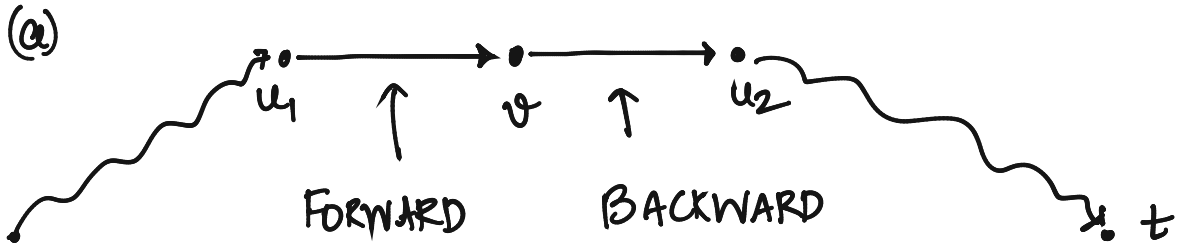

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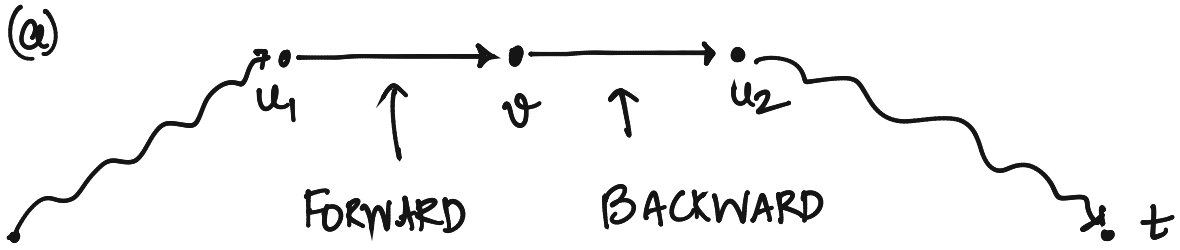

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MAX-FLOW (G)
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MAX-FLOW (G) ALL CAPCITIES ARE INTEGERS.

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LEMMA: AFTER EACH ITERATION OF WHILE LOOP, $f(e)$ IS AN INTEGER $\forall e \in E$.

LEMMA: LET f BE A FLOW AT THE START OF WHILE LOOP & f' AT THE END OF WHILE LOOP, THEN

$$\sum_{e \text{ out of } s} f'(e) > \sum_{e \text{ out of } s} f(e)$$

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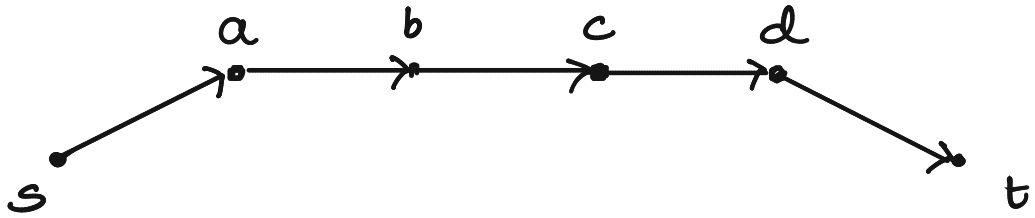

SIMPLE PATH $\Rightarrow$ EACH VERTEX IS VISITED ONCE.

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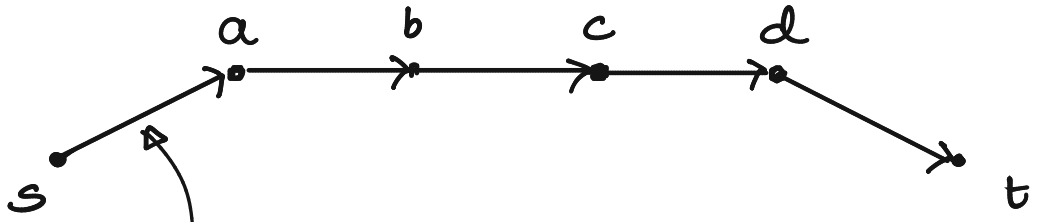

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Q: IS THIS A FORWARD OR BACKWARD EDGE ?

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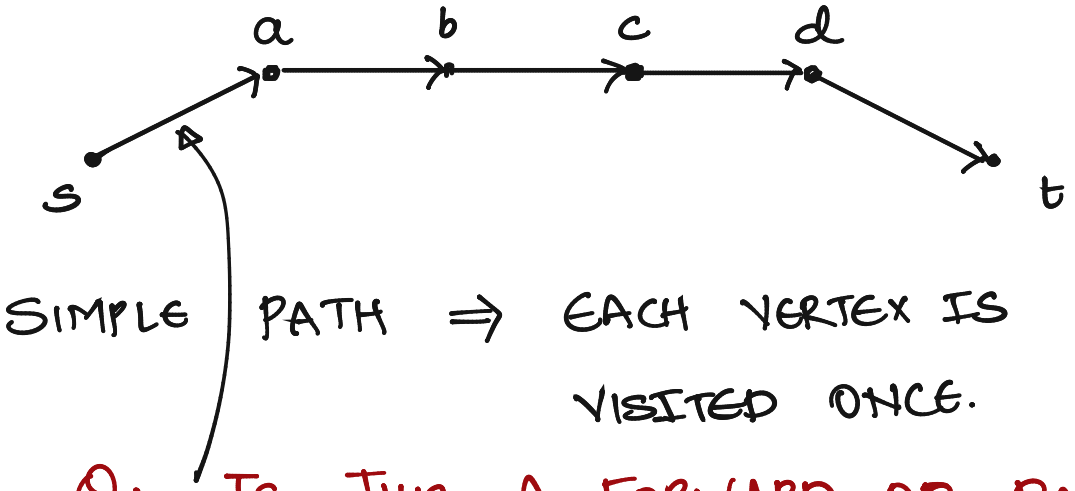

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$$\Rightarrow \sum_{e \text{ OUT OF } s} f'(e) = \sum_{e \text{ OUT of } s} f(e) + b$$

$$\Rightarrow \text{FLOW OUT of } s \text{ IN } f' > \text{FLOW OUT OF } s \text{ IN } f.$$

OBSERVATION: FLOW OUT OF S INCREASES BY ATLEAST 1 AFTER EACH ITERATION

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O(m+n)  { FIND GR FOR THE NEW FLOW f
        }
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```

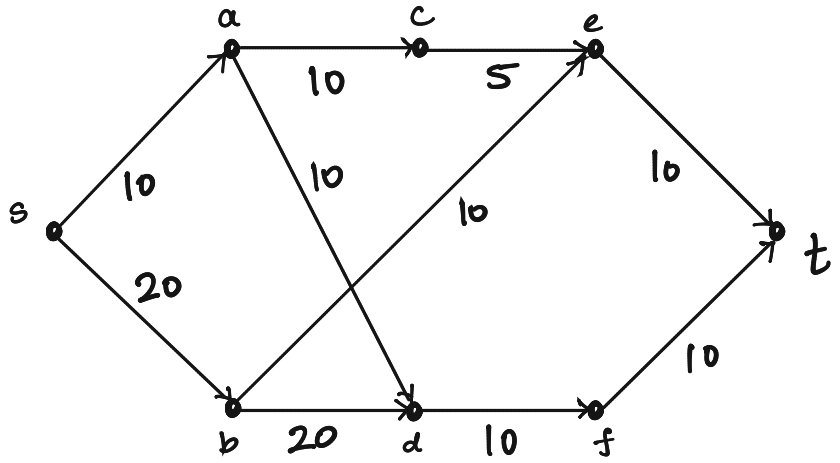
RUNNING TIME : $O((m+n)C)$.

$= O(mC)$

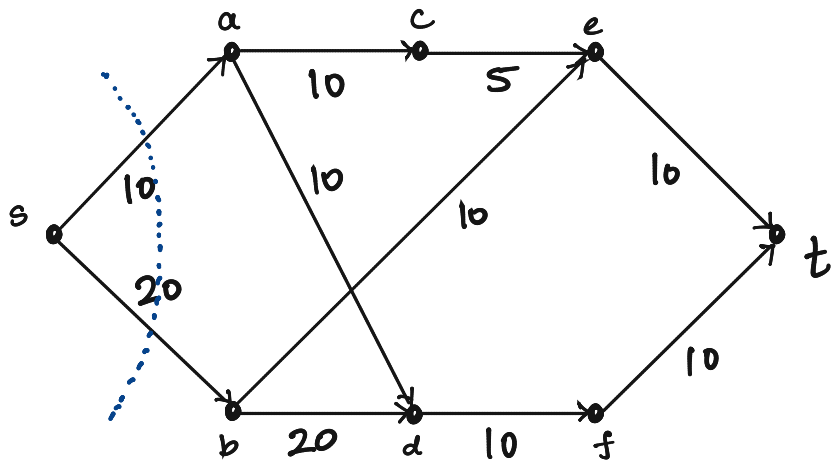
SHOWN THAT THE ALGORITHM TERMINATES AND RETURNS A VALID FLOW IN $O(mC)$ TIME.

FOR CORRECTNESS, WE STILL NEED TO SHOW THAT WE RETURN A MAXIMUM FLOW.

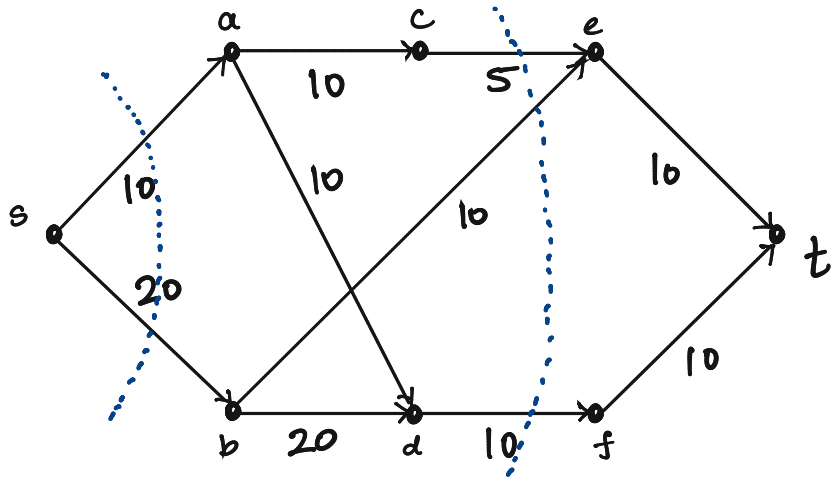
IN FACT THIS ALGORITHM IS FAMOUS DUE TO ITS CORRECTNESS ANALYSIS.



FLOW FROM s to t IS ALWAYS ≤

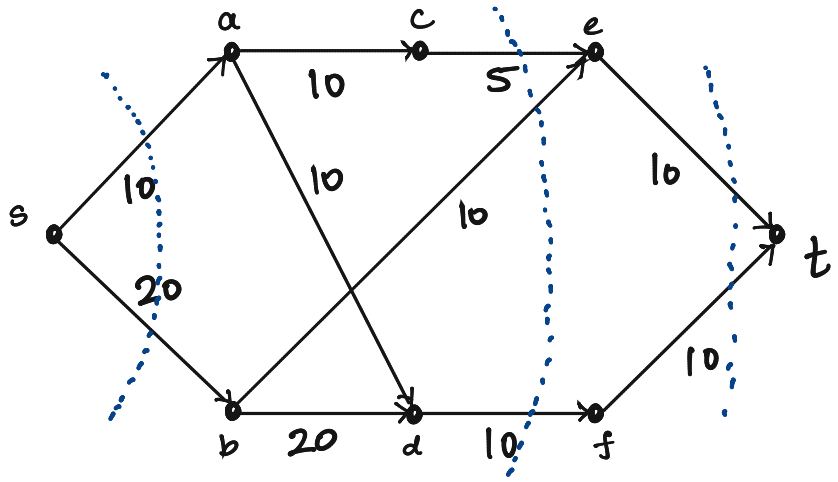


FLOW FROM s to t IS ALWAYS $\leq$ 30.



FLOW FROM s to t IS ALWAYS $\leq$ 30.

FLOW FROM s to t $\leq$ 25



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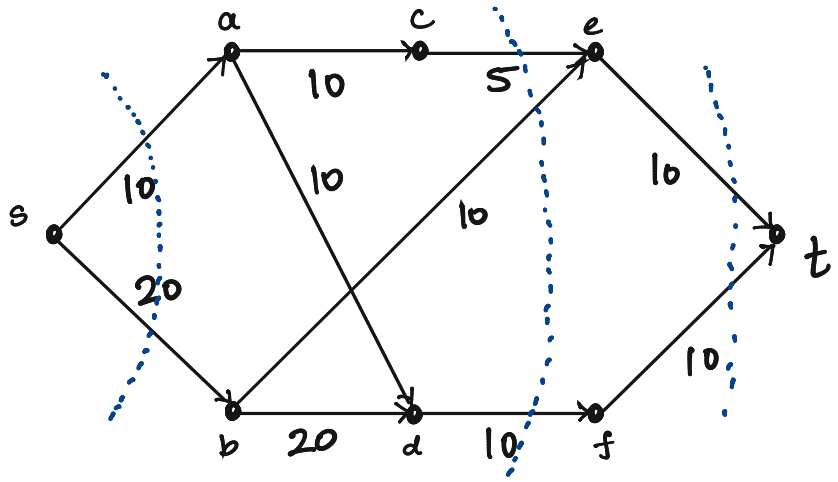
FLOW FROM s to t ≤ 25

FLOW FROM s to t ≤ 20

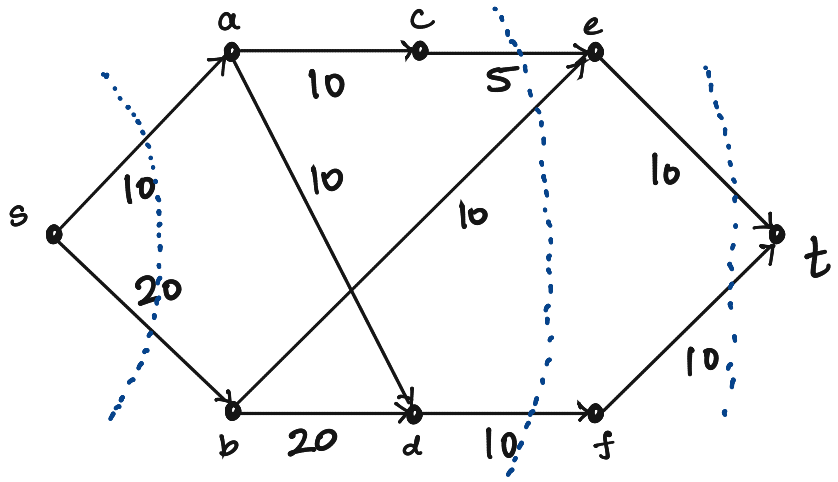


FLOW FROM s to t IS ALWAYS $\leq$ 30.

$A = \{s\}$

$B = \{a, b, c, d, e, f, t\}$

$c(A,B) = \{(s,a), (s,b)\}$



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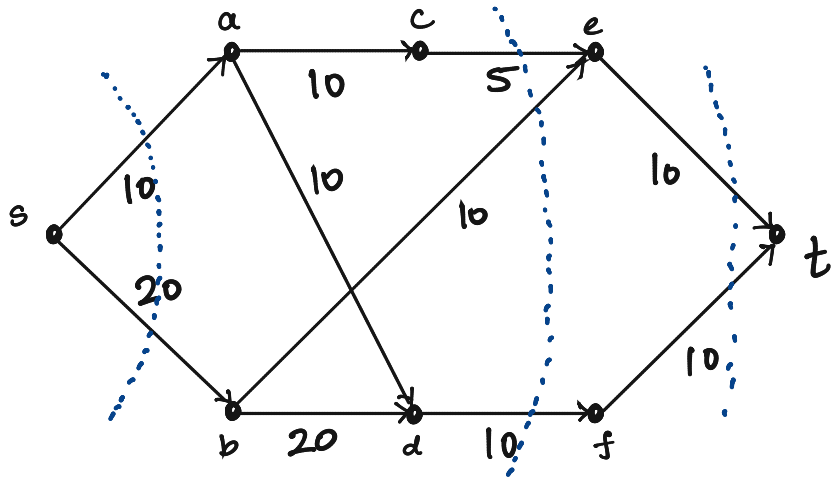

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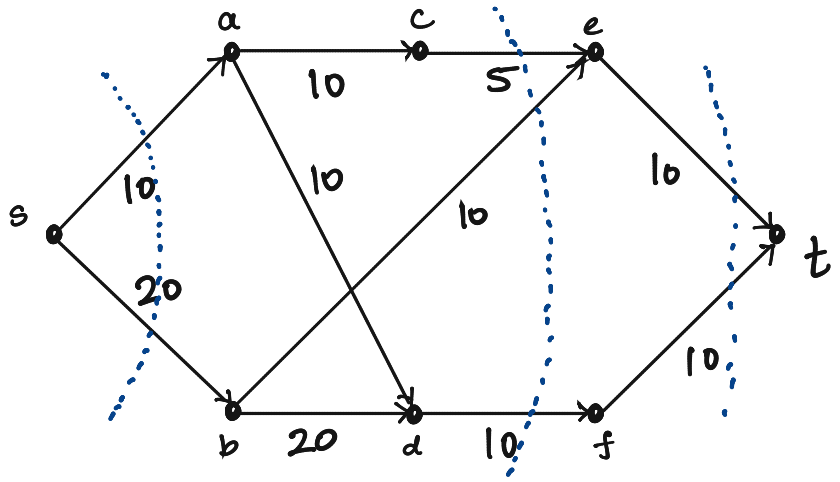
FLOW FROM s to t $\leq$ 25

FLOW FROM s to t $\leq$ 20

$A = \{ s, a, b, c, d, e, f \}$

$B = \{ t \}$

$c(A,B) = \{ (e,t), (f,t) \}$.



FLOW FROM s to t IS ALWAYS $\leq$ 30.

FLOW FROM s to t $\leq$ 25

FLOW FROM s to t $\leq$ 20

A = SET CONTAINING s

B = SET CONTAINING t

$C(A,B) = \{(u,v) \mid u \in A \text{ & } v \in B\}$

LEMMA: GIVEN A CUT $c(A,B)$, THE FLOW FROM s to t $\leq \sum_{e \in C(A,B)} w(e)$

LEMMA : LET f BE A st FLOW AND $c(A,B)$ BE st CUT, THEN

$$\text{FLOW FROM } s \text{ to } t = \sum_{\substack{e \text{ OUT} \\ \text{of } A}} f(e) - \sum_{\substack{e \text{ IN} \\ A}} f(e)$$

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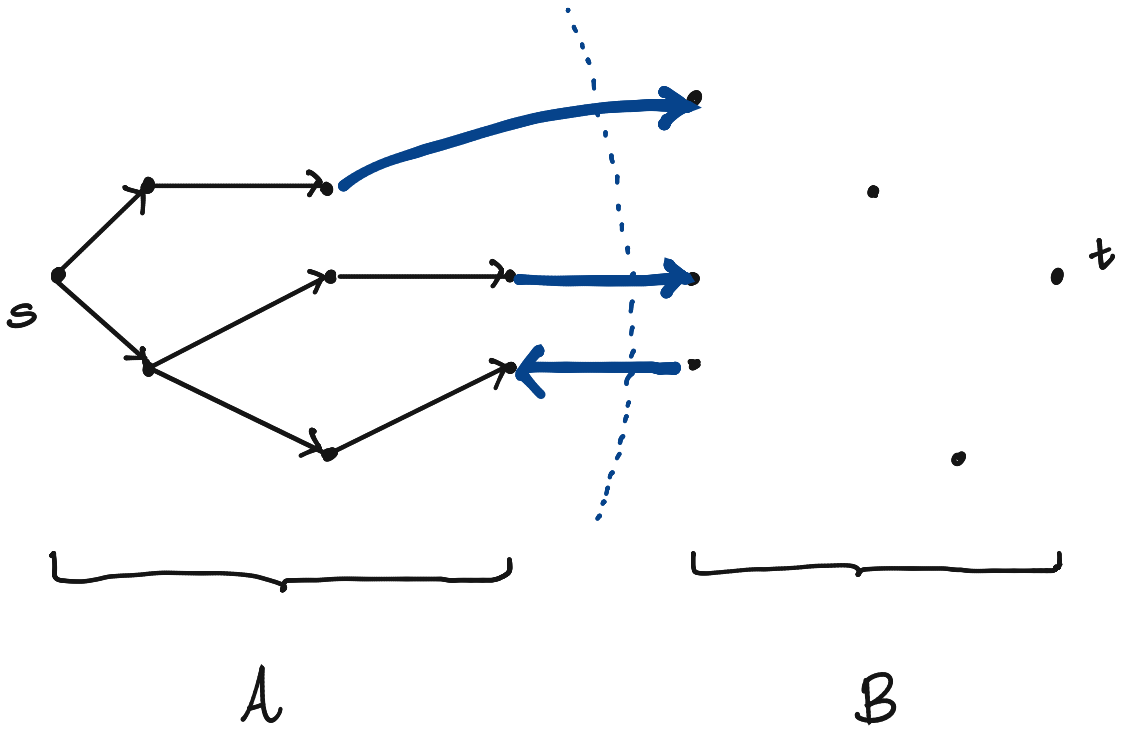
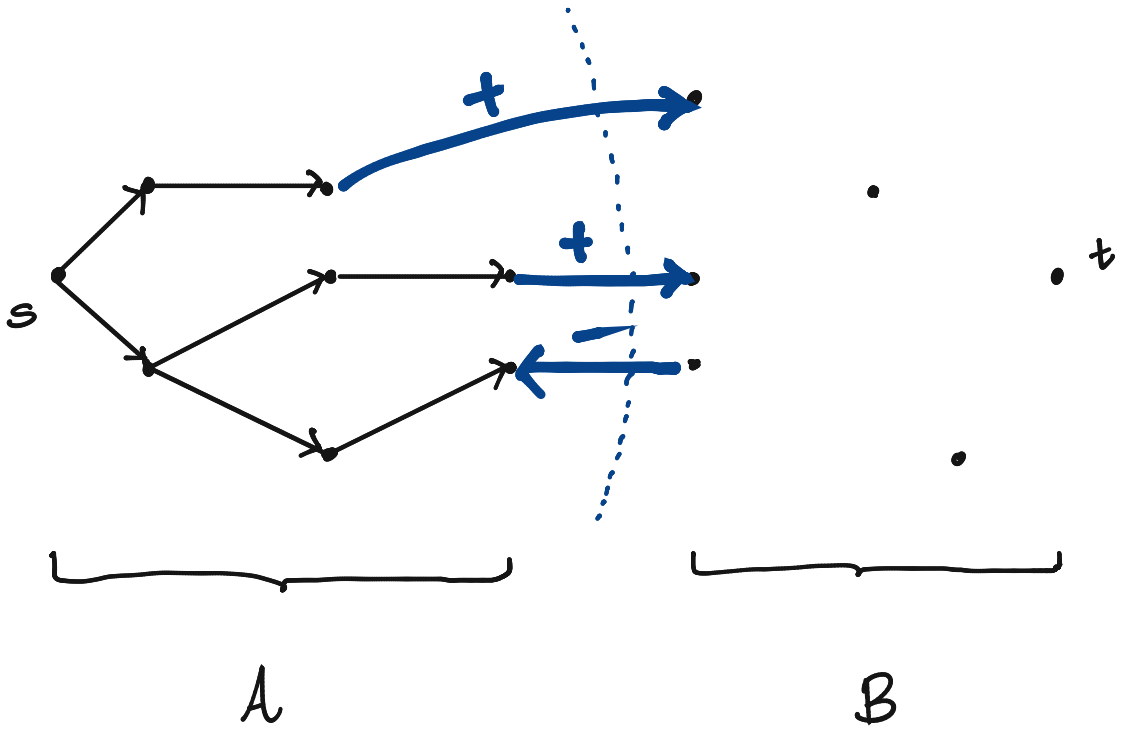


LEMMA: LET f BE A st FLOW AND $c(A,B)$ BE st CUT, THEN

$$\text{FLOW FROM } s \text{ to } t = \sum_{e \text{ OUT of } A} f(e) - \sum_{e \text{ IN } A} f(e)$$



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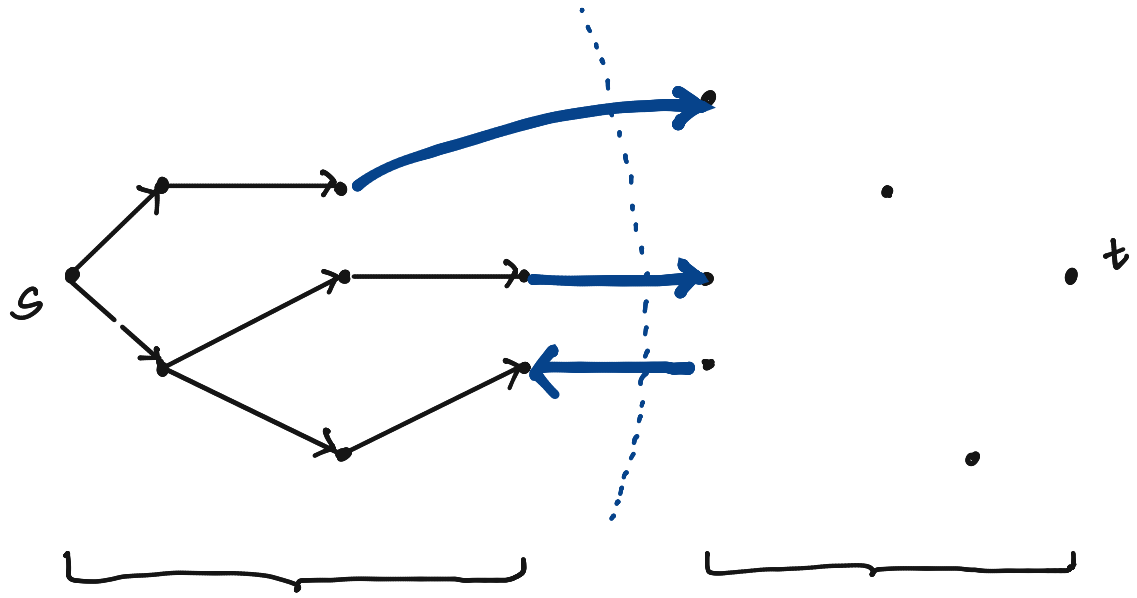

$$\sum_{e \text{ OUT OF } s} f(e) - \sum_{e \text{ IN } s} f(e)$$

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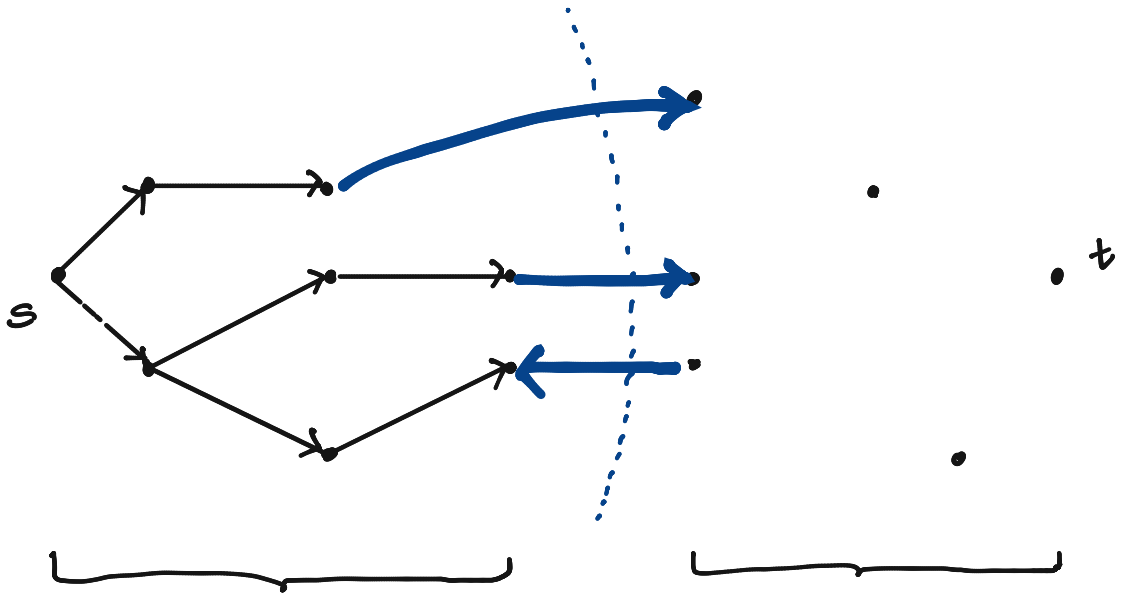
$$\text{FLOW FROM } s \text{ to } t = \sum_{e \text{ OUT of } A} f(e) - \sum_{e \text{ IN } A} f(e)$$



$$\sum_{e \text{ OUT OF } s} f(e) - \sum_{e \text{ IN } s} f(e) = \text{FLOW FROM } s \text{ to } t$$

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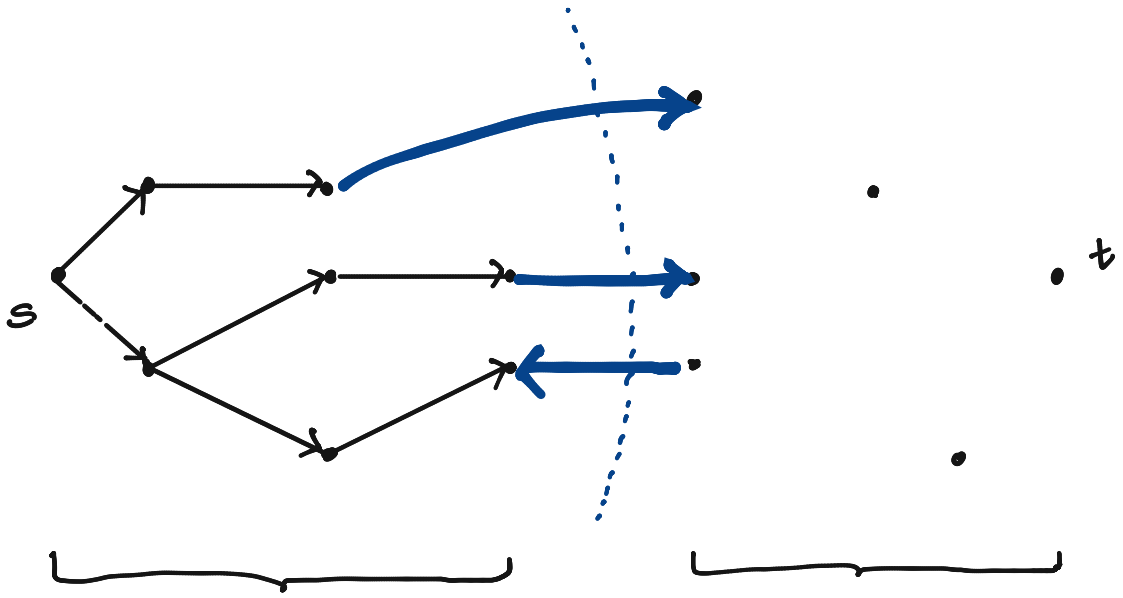


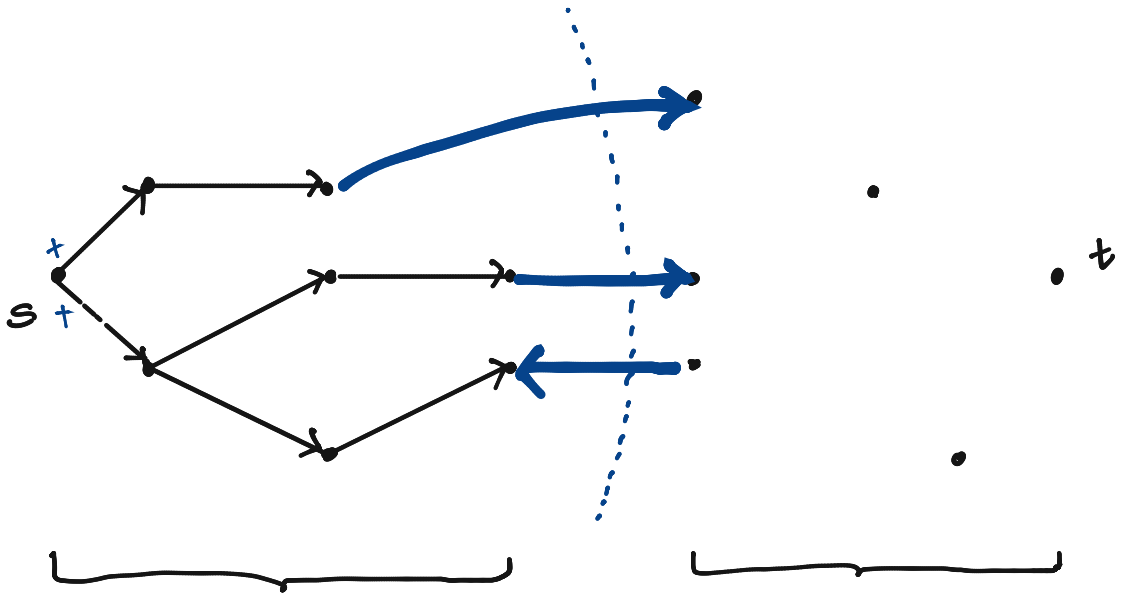
$$\sum_{e \text{ OUT OF } s} f(e) - \sum_{e \text{ IN } s} f(e) = \text{FLOW FROM } s \text{ to } t$$

FOR ANY OTHER INTERNAL NODE v IN A,

$$\sum_{e \text{ OUT OF } v} f(e) - \sum_{e \text{ IN } v} f(e)$$

LEMMA: LET f BE A st FLOW AND $c(A,B)$ BE st CUT, THEN

FLOW FROM s to t $= \sum_{\substack{e \text{ OUT} \\ \text{of } A}} f(e) - \sum_{\substack{e \text{ IN} \\ A}} f(e)$



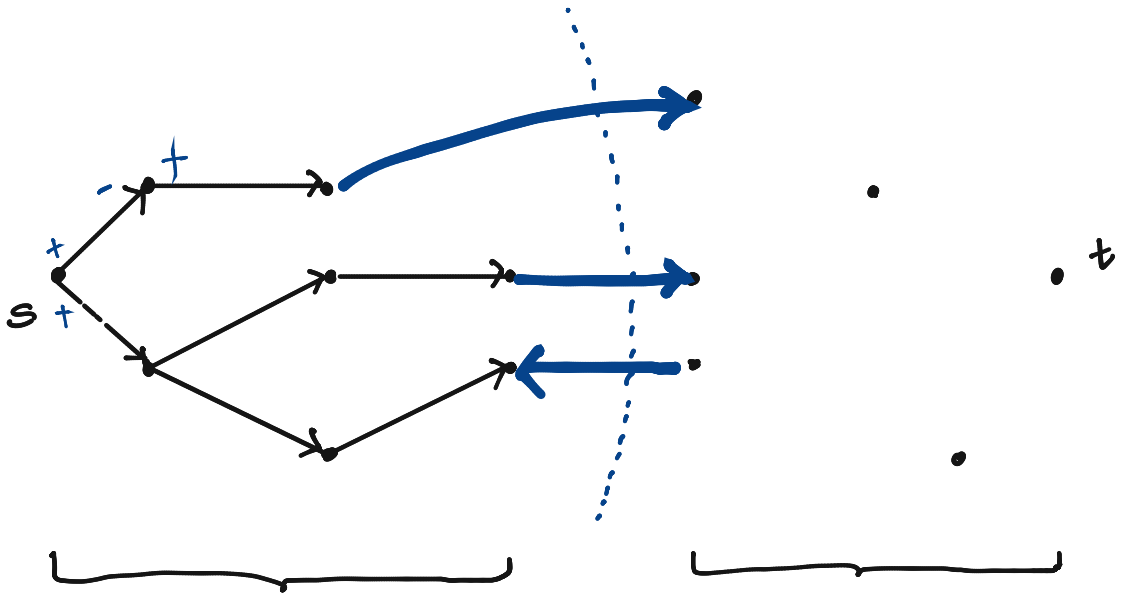
$$\sum_{\substack{e \text{ OUT} \\ \text{OF } s}} f(e) - \sum_{\substack{e \text{ IN} \\ s}} f(e) = \text{FLOW FROM } s \text{ to } t$$

FOR ANY OTHER INTERNAL NODE v IN A,

$$\sum_{\substack{e \text{ OUT} \\ \text{OF } v}} f(e) - \sum_{\substack{e \text{ IN} \\ v}} f(e) = 0$$

LEMMA: LET f BE A st FLOW AND $c(A,B)$ BE st CUT, THEN

FLOW FROM s to t $= \sum_{e \text{ OUT of } A} f(e) - \sum_{e \text{ IN } A} f(e)$



$$\sum_{e \text{ OUT OF } s} f(e) - \sum_{e \text{ IN } s} f(e) = \text{FLOW FROM } s \text{ to } t$$

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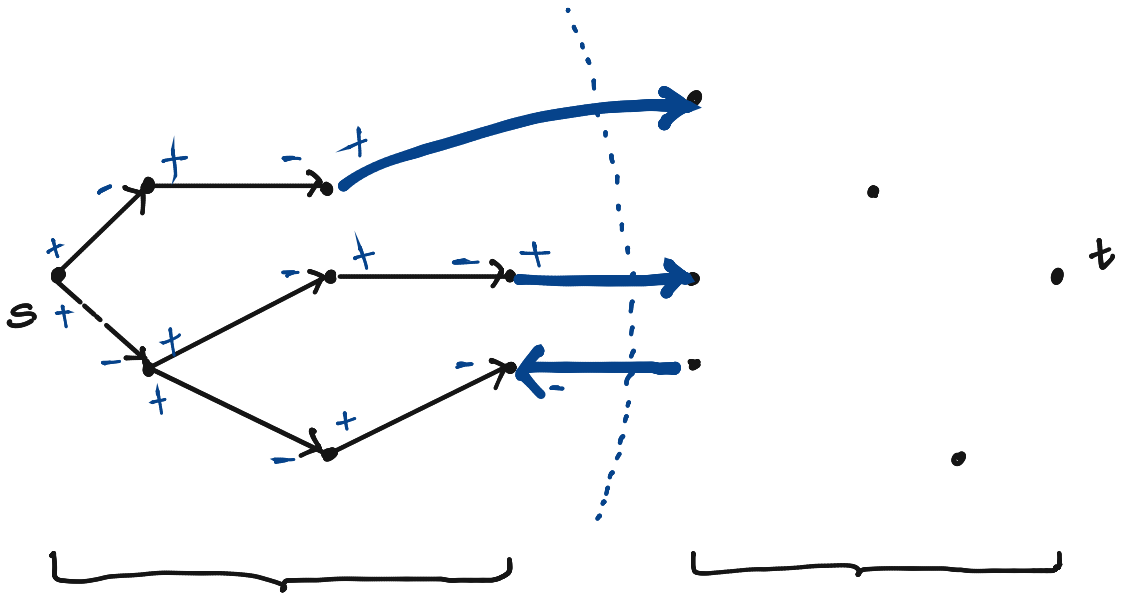
FLOW FROM s to t $= \sum_{e \text{ OUT of } A} f(e) - \sum_{e \text{ IN } A} f(e)$



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FOR ANY OTHER INTERNAL NODE v IN A,

$$\sum_{e \text{ OUT OF } v} f(e) - \sum_{e \text{ IN } v} f(e) = 0$$

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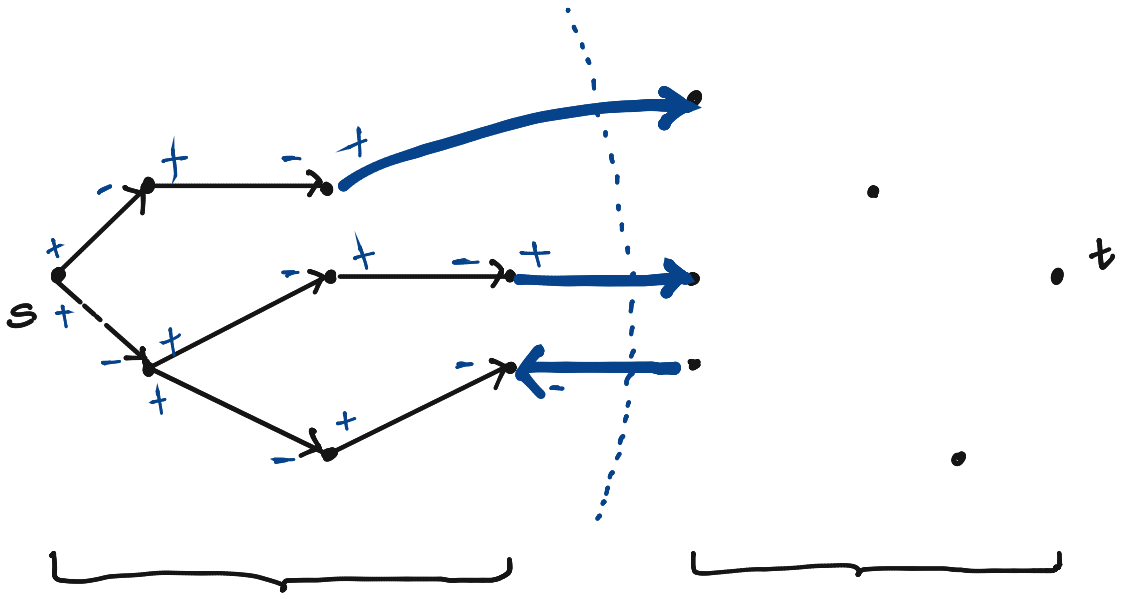

$$\sum_{e \text{ OUT OF } s} f(e) - \sum_{e \text{ IN } s} f(e) = \text{FLOW FROM } s \text{ to } t$$

FOR ANY OTHER INTERNAL NODE v IN A,

$$\sum_{e \text{ OUT OF } v} f(e) - \sum_{e \text{ IN } v} f(e) = 0$$

$$\sum_{e \text{ OUT of } A} f(e) - \sum_{e \text{ IN } A} f(e) = \text{FLOW FROM } s \text{ to } t$$

→ NOW LET US LOOK AT OUR ALGO.

→ AFTER SOME ITERATION, WE ARE LEFT WITH A RESIDUAL GRAPH SUCH THAT THERE IS NO st PATH IN G_R.

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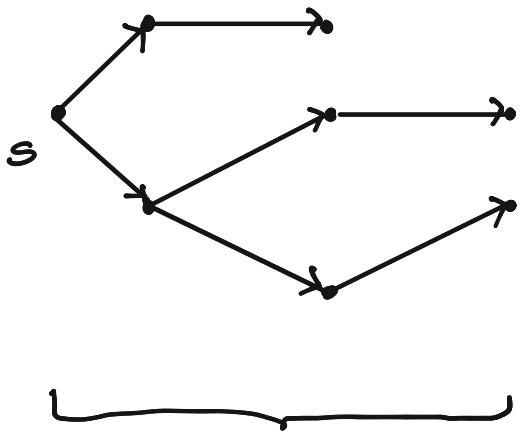
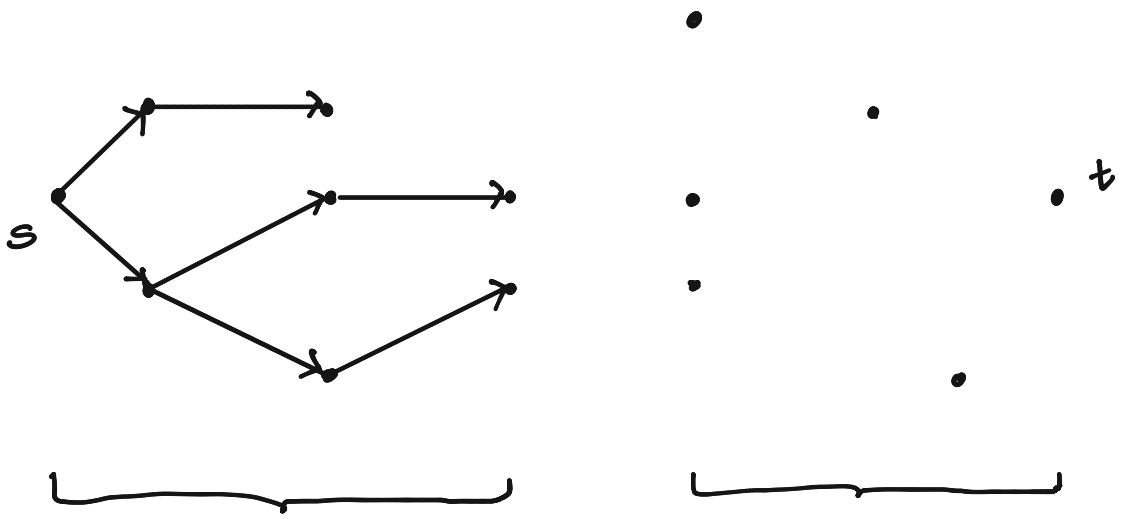


VERTICES CONNECTED TO s IN G_R

→ NOW LET US LOOK AT OUR ALGO.

→ AFTER SOME ITERATION, WE ARE LEFT WITH A RESIDUAL GRAPH SUCH THAT THERE IS NO st PATH IN G_R.

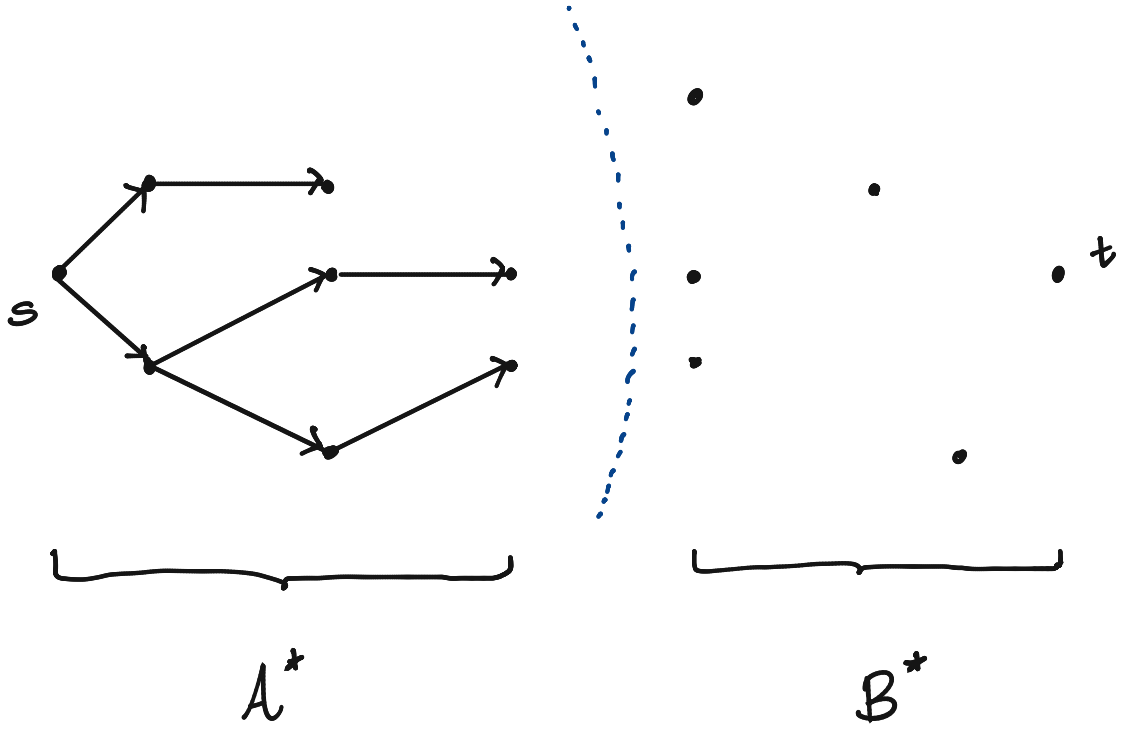


VERTICES CONNECTED TO s IN G_R

VERTICES NOT CONNECTED TO s IN G_R

→ NOW LET US LOOK AT OUR ALGO.

→ AFTER SOME ITERATION, WE ARE LEFT WITH A RESIDUAL GRAPH SUCH THAT THERE IS NO st PATH IN G_R.

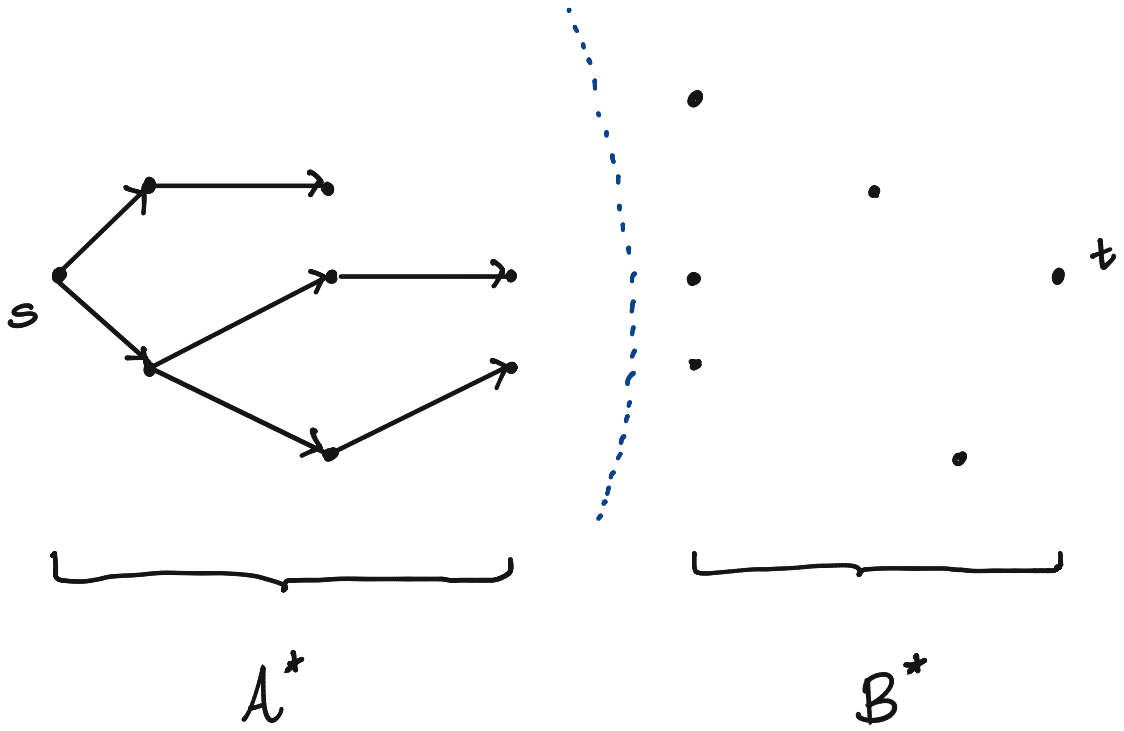


BY OUR LEMMA, FLOW FROM s to t

$\leq$

→ NOW LET US LOOK AT OUR ALGO.

→ AFTER SOME ITERATION, WE ARE LEFT WITH A RESIDUAL GRAPH SUCH THAT THERE IS NO st PATH IN G_R.

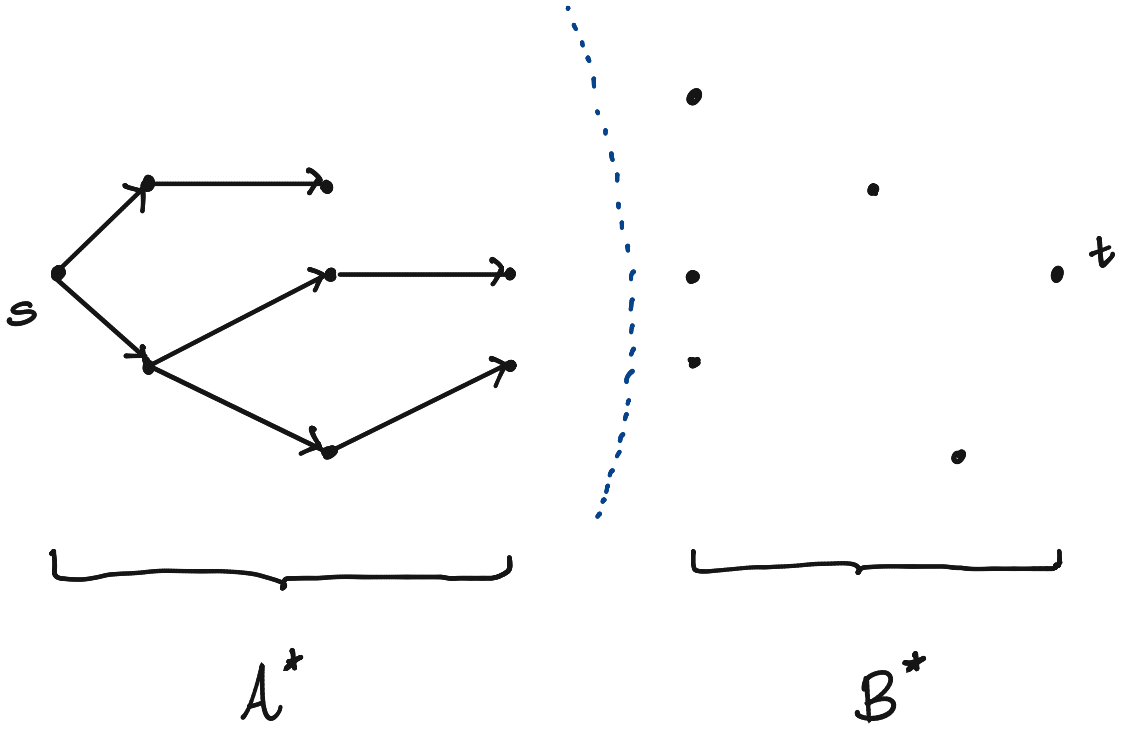


BY OUR LEMMA, FLOW FROM s to t

$\leq$ WEIGHT OF CUT $c(A^*, B^*)$.

→ NOW LET US LOOK AT OUR ALGO.

→ AFTER SOME ITERATION, WE ARE LEFT WITH A RESIDUAL GRAPH SUCH THAT THERE IS NO st PATH IN G_R.



BY OUR LEMMA, FLOW FROM s to t $\leq$ WEIGHT OF CUT $c(A^*, B^*)$.

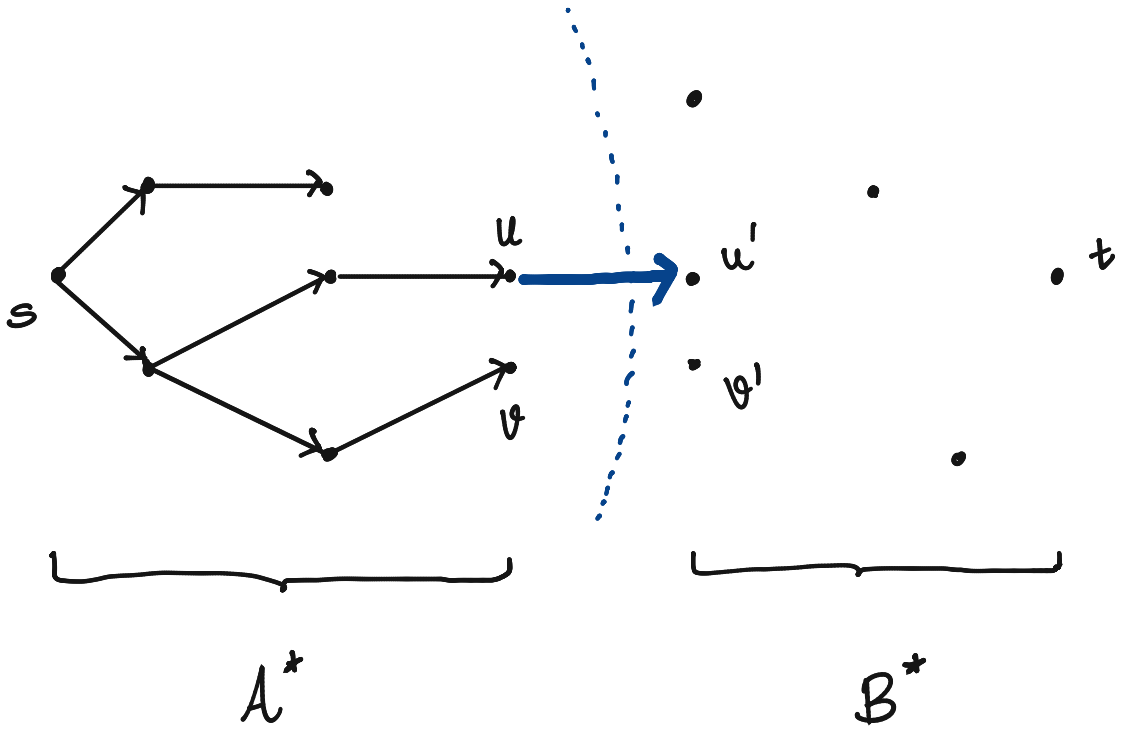
WE WILL PROVE

FLOW FROM s to t = WT. OF MINIMUM CUT

$c(A^*, B^*)$ IS THE MINIMUM st CUT.

→ NOW LET US LOOK AT OUR ALGO.

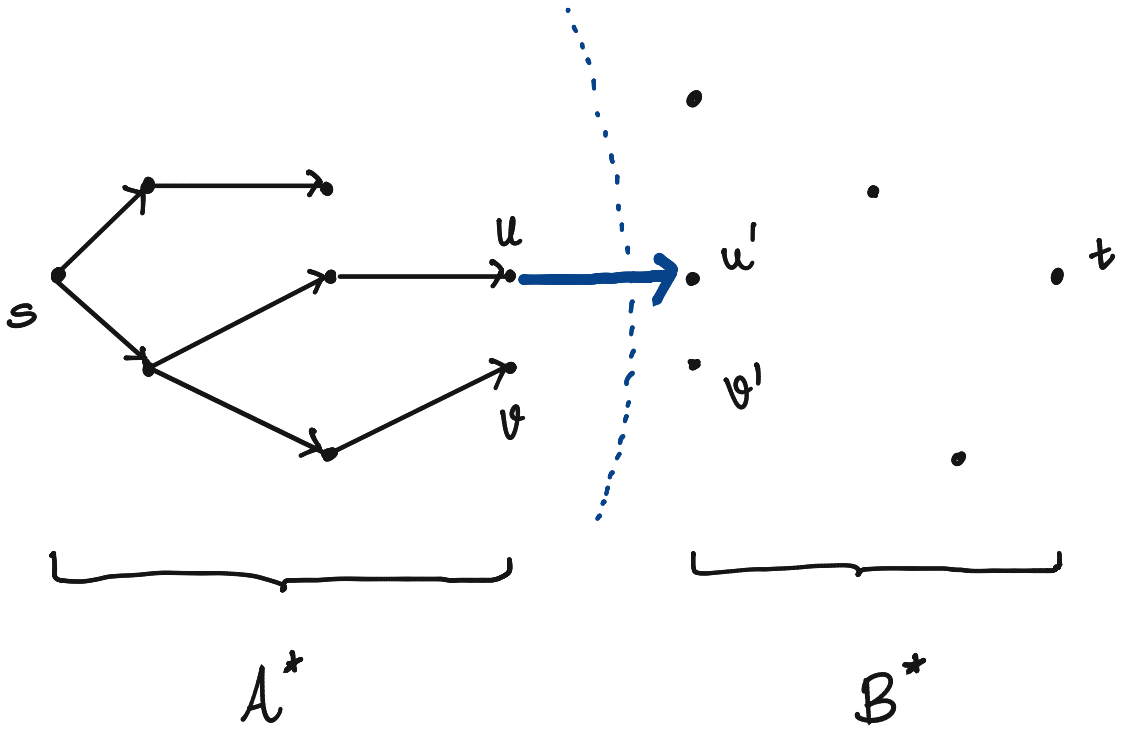
→ AFTER SOME ITERATION, WE ARE LEFT WITH A RESIDUAL GRAPH SUCH THAT THERE IS NO st PATH IN G_R.



Q WHAT IS THE FLOW THROUGH (u,u') AT THE END OF OUR ALGO ?

→ NOW LET US LOOK AT OUR ALGO.

→ AFTER SOME ITERATION, WE ARE LEFT WITH A RESIDUAL GRAPH SUCH THAT THERE IS NO st PATH IN G_R.



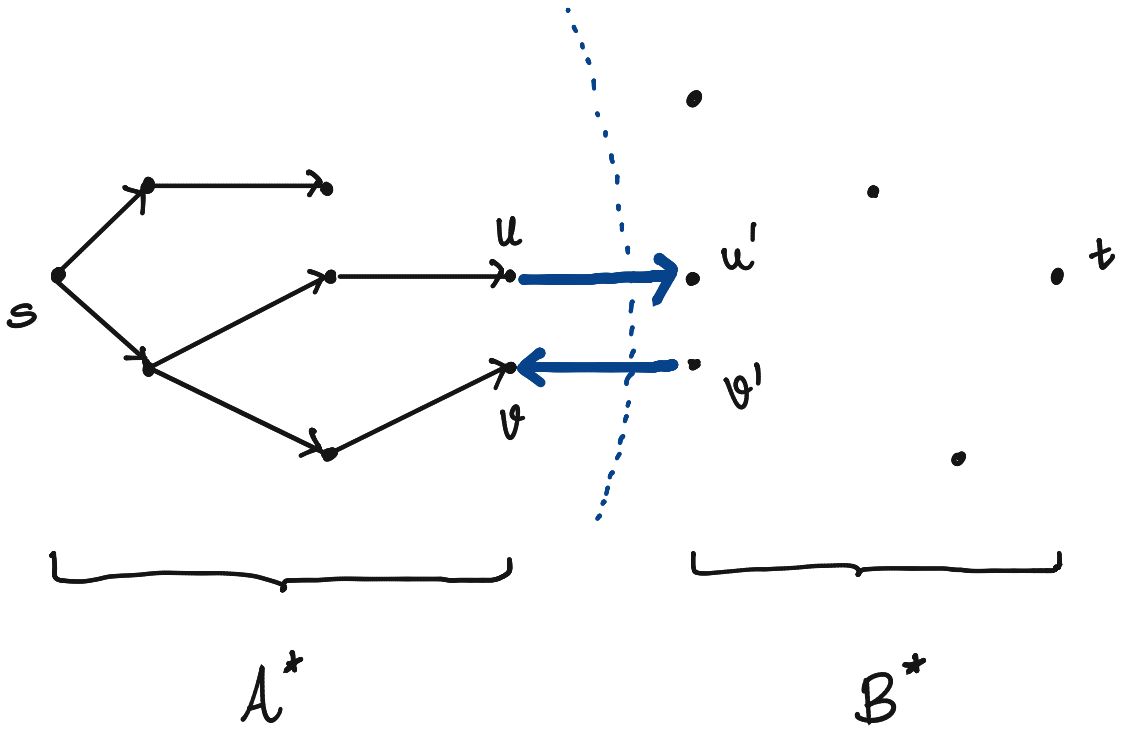
Q WHAT IS THE FLOW THROUGH (u,u') AT THE END OF OUR ALGO ?

A (u,u') IS SATURATED

$f(u,u') = c(u,u')$

→ NOW LET US LOOK AT OUR ALGO.

→ AFTER SOME ITERATION, WE ARE LEFT WITH A RESIDUAL GRAPH SUCH THAT THERE IS NO st PATH IN G_R.

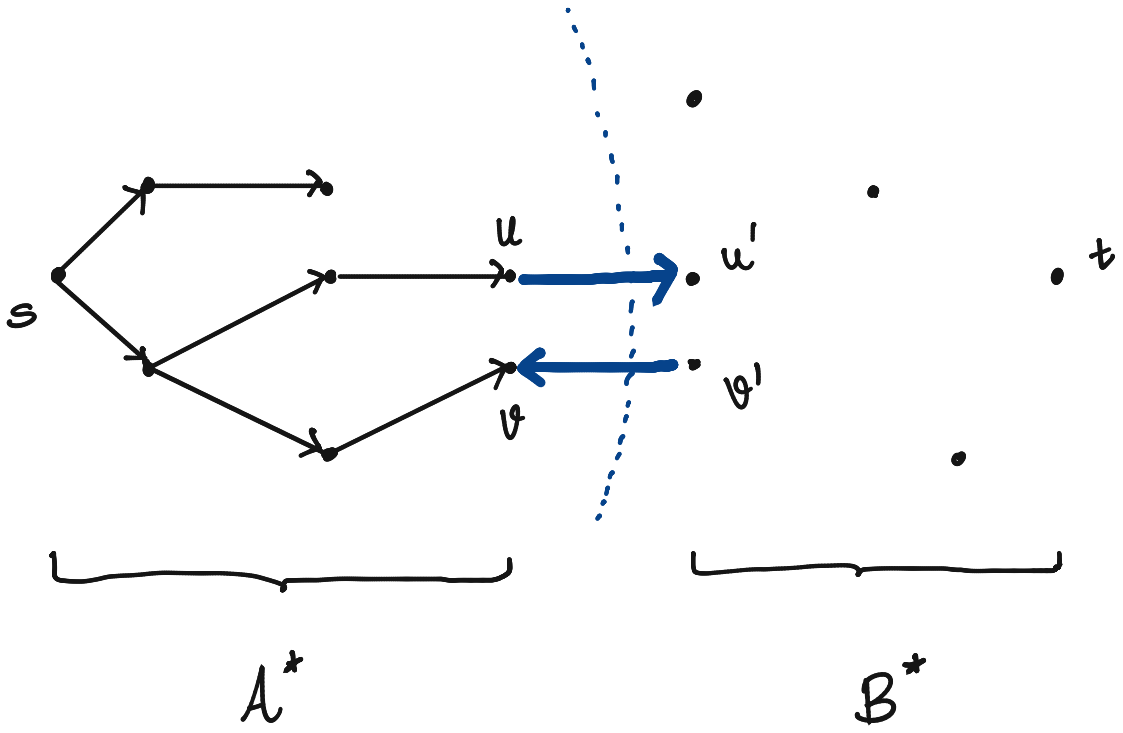


Q WHAT IS THE FLOW THROUGH (v, v') AT THE END OF OUR ALGO ?

A

→ NOW LET US LOOK AT OUR ALGO.

→ AFTER SOME ITERATION, WE ARE LEFT WITH A RESIDUAL GRAPH SUCH THAT THERE IS NO st PATH IN G_R.

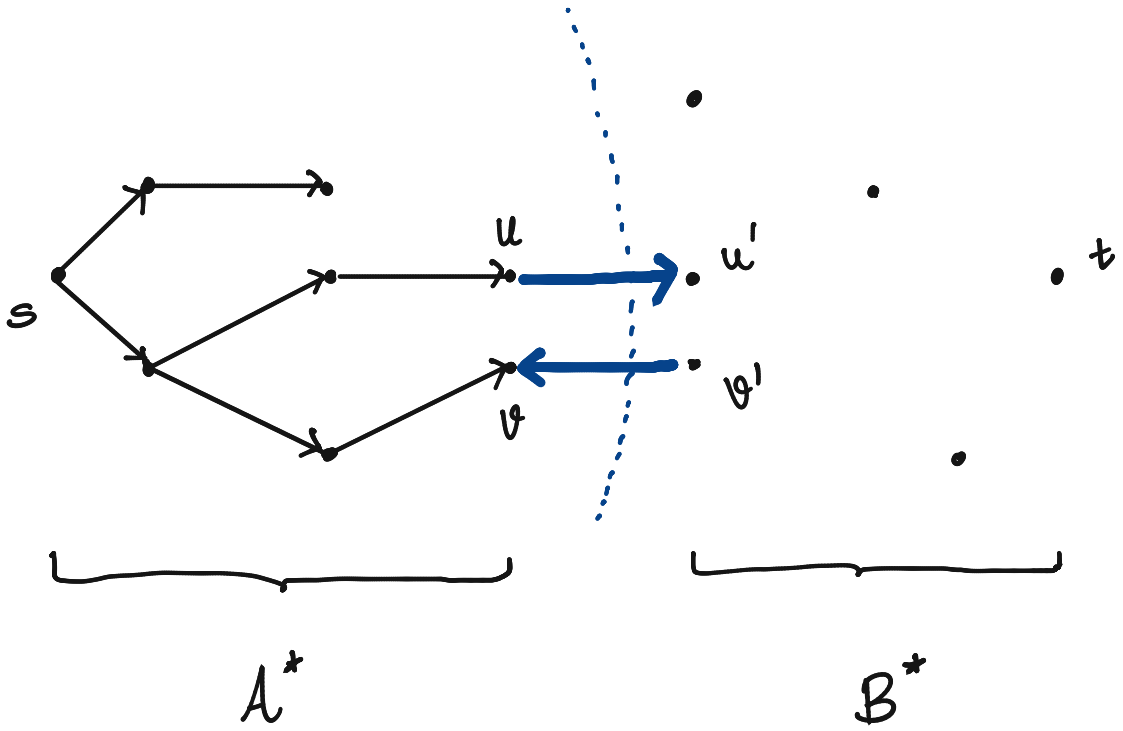


Q WHAT IS THE FLOW THROUGH (v, v') AT THE END OF OUR ALGO ?

A · $f(v, v') = 0$

→ NOW LET US LOOK AT OUR ALGO.

→ AFTER SOME ITERATION, WE ARE LEFT WITH A RESIDUAL GRAPH SUCH THAT THERE IS NO st PATH IN G_R.



FLOW FROM s to t $= \sum_{e \text{ OUT OF } A^*} f(e) - \sum_{e \text{ IN } A^*} f(e)$

→ NOW LET US LOOK AT OUR ALGO.

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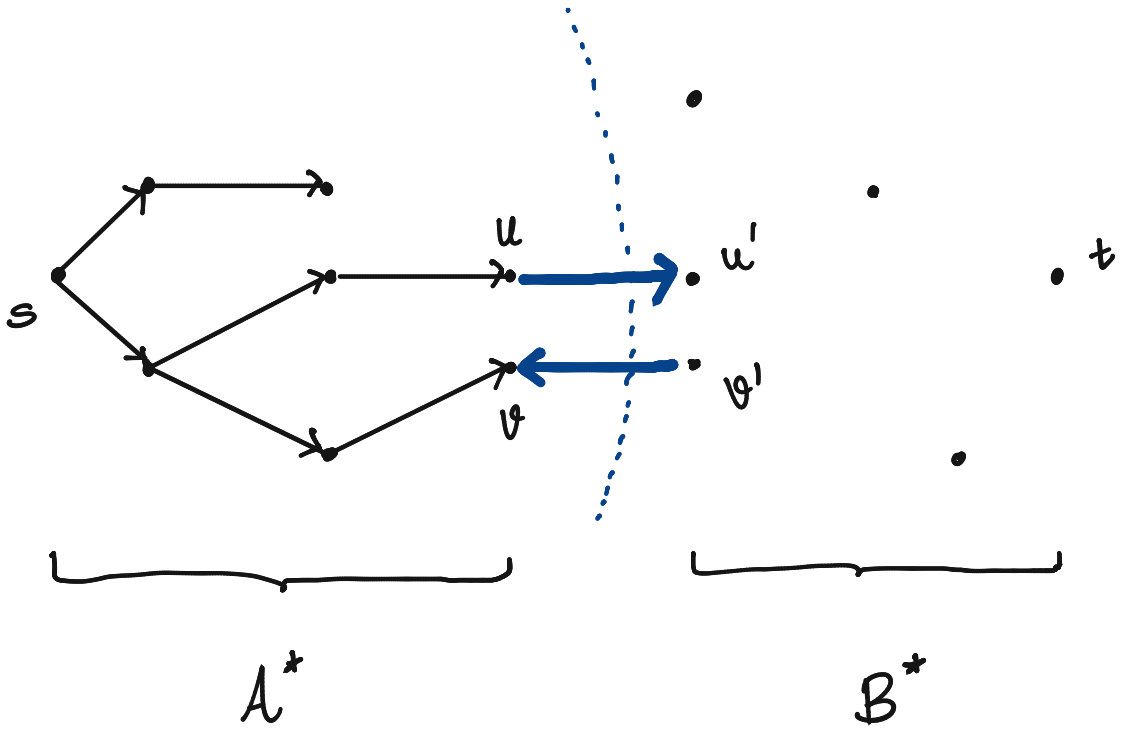


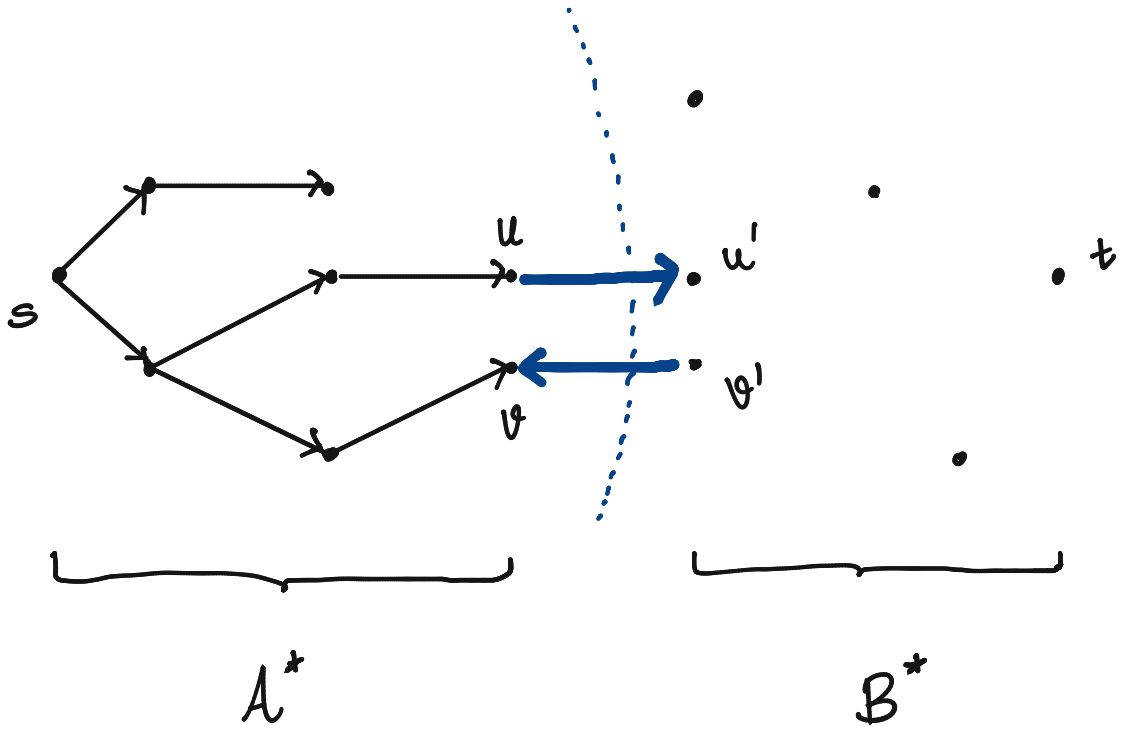
FLOW FROM s to t $= \sum_{e \text{ OUT OF } A^*} f(e) - \sum_{e \text{ IN } A^*} f(e)$

$= \sum_{e \text{ out of } A^*} c(e)$

= WEIGHT OF $c(A^*, B^*)$

→ NOW LET US LOOK AT OUR ALGO.

→ AFTER SOME ITERATION, WE ARE LEFT WITH A RESIDUAL GRAPH SUCH THAT THERE IS NO st PATH IN G_R.



FLOW FROM s to t = WEIGHT OF CUT $c(A^*, B^*)$

BUT FLOW FROM s to t $\leq$ WEIGHT OF ANY CUT $c(A, B)$

$\Rightarrow c(A^*, B^*)$ IS THE MINIMUM WEIGHT s-t CUT.

→ ANY s-t FLOW ≤ WEIGHT OF MINIMUM st CUT

→ OUR ALGORITHM RETURNS A FLOW = WEIGHT OF MINIMUM st CUT

⇒ OUR ALGORITHM RETURNS THE MAXIMUM FLOW

→ ANY s-t FLOW ≤ WEIGHT OF MINIMUM st CUT

→ OUR ALGORITHM RETURNS A FLOW = WEIGHT OF MINIMUM st CUT

⇒ OUR ALGORITHM RETURNS THE MAXIMUM FLOW

FORD - FULKERSON ALGORITHM.