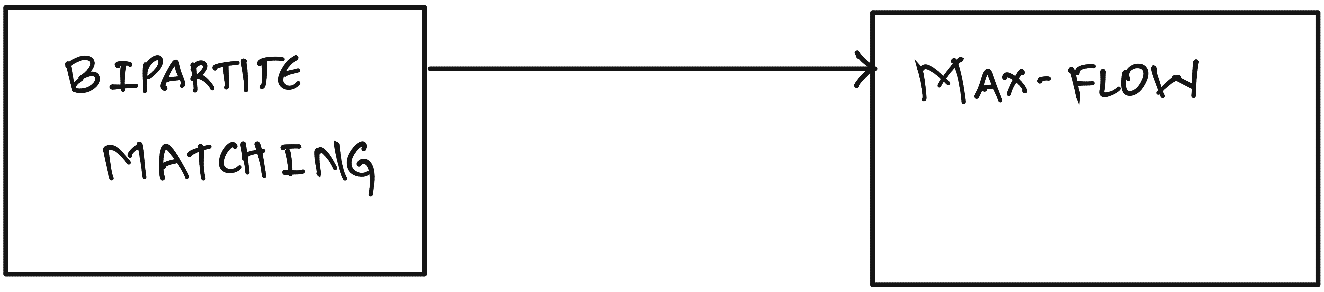
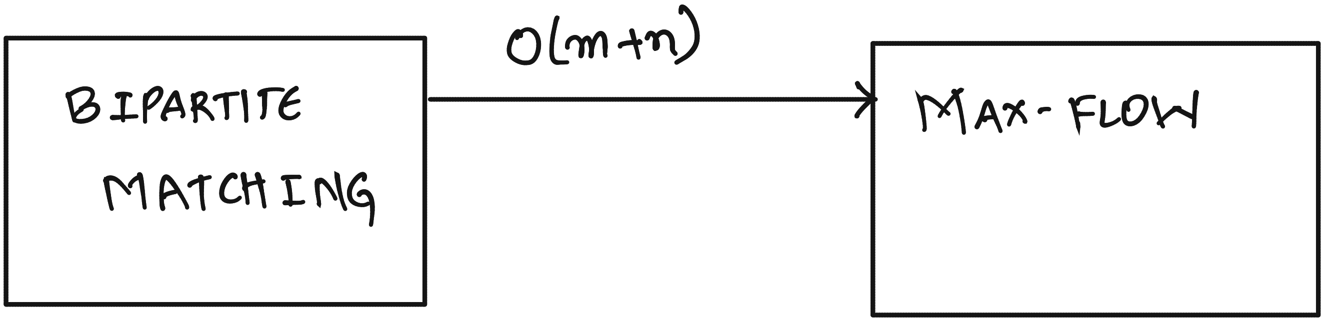
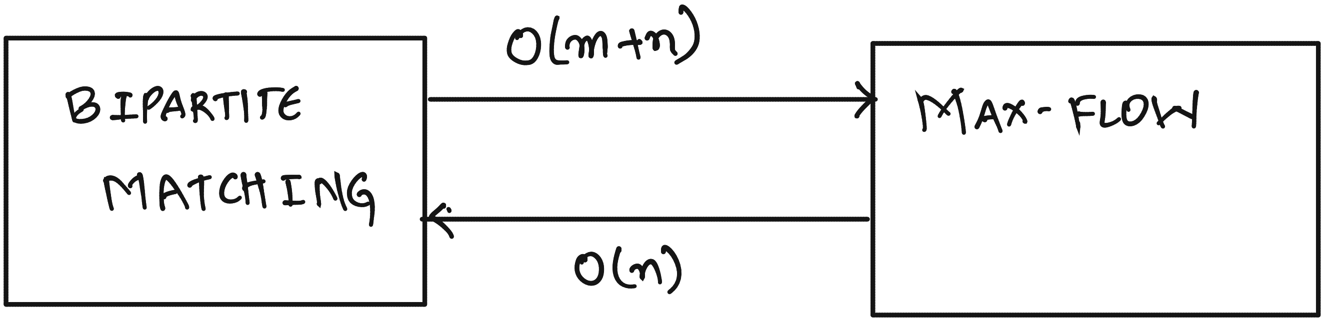
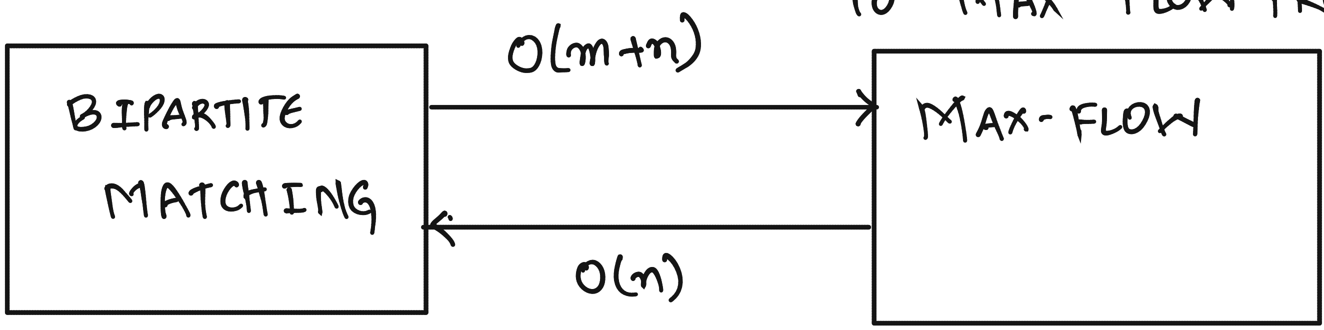
BIPARTITE MATCHING


BIPARTITE
MATCHING
MAX-FLOW


BIPARTITE MATCHING
O(m+n)
MAX-FLOW

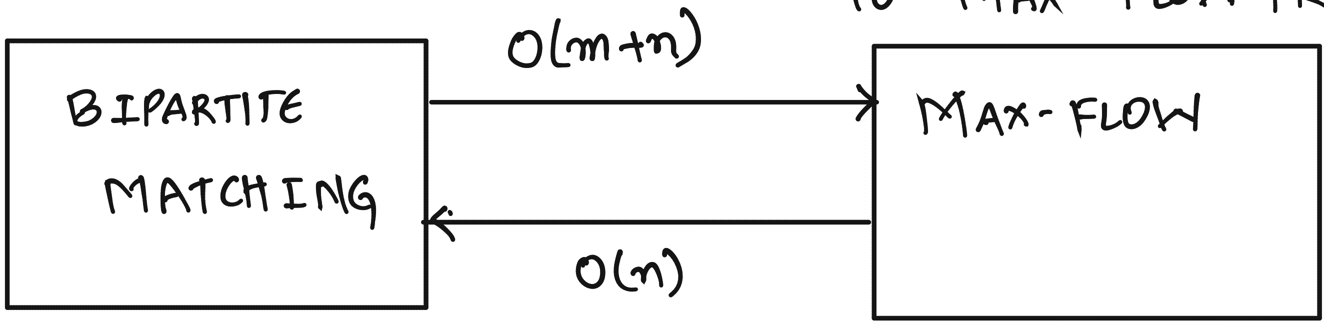

BIPARTITE MATCHING
O(m+n)
MAX-FLOW
O(n)

In Polynomial Time We Reduced Max Match Problem to Max Flow Problem.



MAX MATCHING $\le_p$ MAXFLOW

In Polynomial Time We Reduced MAX MATCH PROBLEM TO MAX FLOW PROBLEM.



MAX MATCHING $\leq_p$ MAXFLOW

IN POLYNOMIAL TIME WE REDUCED MAX MATCH PROBLEM TO MAX FLOW PROBLEM.

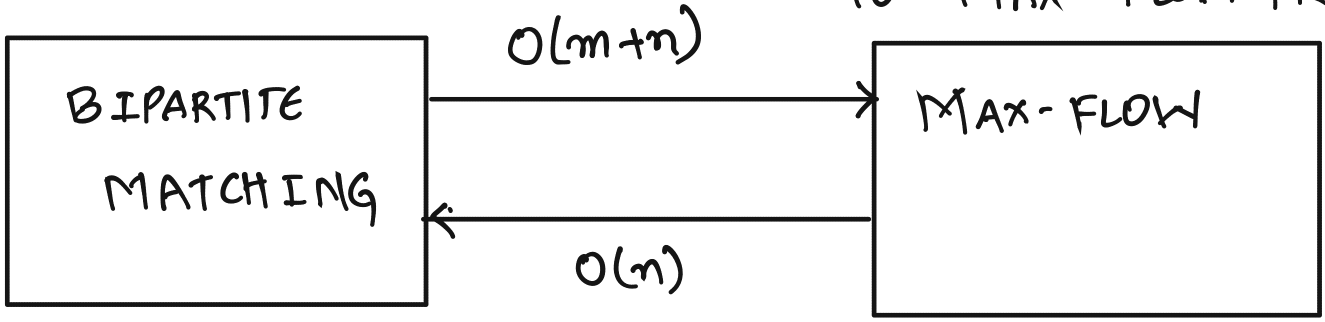
$$
\begin{tikzcd}
\text{BIPARTITE MATCHING} \arrow[r, shift left, "O(m+n)"] & \text{MAX-FLOW} \arrow[l, shift left, "O(n)"]
\end{tikzcd}
$$

CIRCULATION WITH DEMAND

MAX FLOW

MAX MATCHING $\leq_p$ MAXFLOW

In Polynomial Time We Reduced MAX MATCH PROBLEM TO MAX FLOW PROBLEM.



CIRCULATION WITH DEMAND $\leq_p$ MAX FLOW

MAX MATCHING $\leq_p$ MAXFLOW

In POLYNOMIAL TIME WE REDUCED MAX MATCH PROBLEM TO MAX FLOW PROBLEM.

$$\boxed{\text{BIPARTITE MATCHING}} \underset{O(n)}{\overset{O(m+n)}{\rightleftarrows}} \boxed{\text{MAX-FLOW}}$$

CIRCULATION WITH DEMAND

$\leq_p$ MAX FLOW

CIRCULATION WITH DEMAND & LOWER BOUND

CIRCULATION WITH DEMAND

MAX MATCHING $\leq_p$ MAXFLOW

In Polynomial Time We Reduced
MAX MATCH PROBLEM
TO MAX FLOW PROBLEM.

$$
\begin{tikzcd}
\boxed{\text{BIPARTITE MATCHING}} \arrow[r, shift left, "O(m+n)"] & \boxed{\text{MAX-FLOW}} \arrow[l, shift left, "O(n)"]
\end{tikzcd}
$$

CIRCULATION WITH DEMAND
$\leq_p$ MAX FLOW

CIRCULATION WITH DEMAND & LOWER BOUND
$\leq_p$ CIRCULATION WITH DEMAND

DEF : $X \leq_p Y \Rightarrow$ ANY INSTANCE OF X CAN BE SOLVED IN SOME POLYNOMIAL TIME + POLYNOMIAL NUMBER OF CALLS TO ANY ARBITRARY INSTANCES OF Y.

DEF: $X \le_p Y \Rightarrow$ ANY INSTANCE OF X CAN BE SOLVED IN SOME POLYNOMIAL TIME + POLYNOMIAL NUMBER OF CALLS TO ANY ARBITRARY INSTANCES OF Y.

IN OUR EXAMPLE OF

MAX MATCH $\le_p$ MAX FLOW

COMPUTATION STEPS $\rightarrow O(m+n)$

NUMBER OF CALLS TO MAX FLOW $\rightarrow 1$.

<u>LEMMA</u>: IF $X \le_p Y$ & $Y \le_p Z$, THEN $X \le_p Z$.

LEMMA: IF $X \le_p Y$ & $Y \le_p Z$, THEN $X \le_p Z$.

TRANSITIVITY PROPERTY

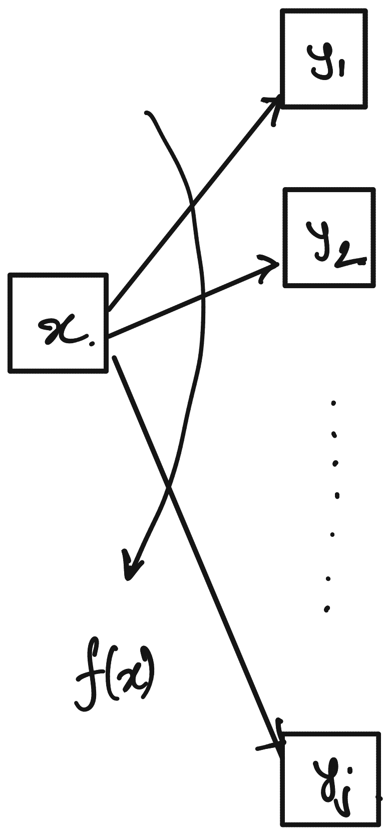
CIRCULATION WITH DEMAND & LOWER BOUND $\le_p$ CIRCULATION WITH DEMAND

CIRCULATION WITH DEMAND $\le_p$ MAX FLOW

$\Rightarrow$ CIRCULATION WITH DEMAND & LOWER BOUND $\le_p$ MAX FLOW.

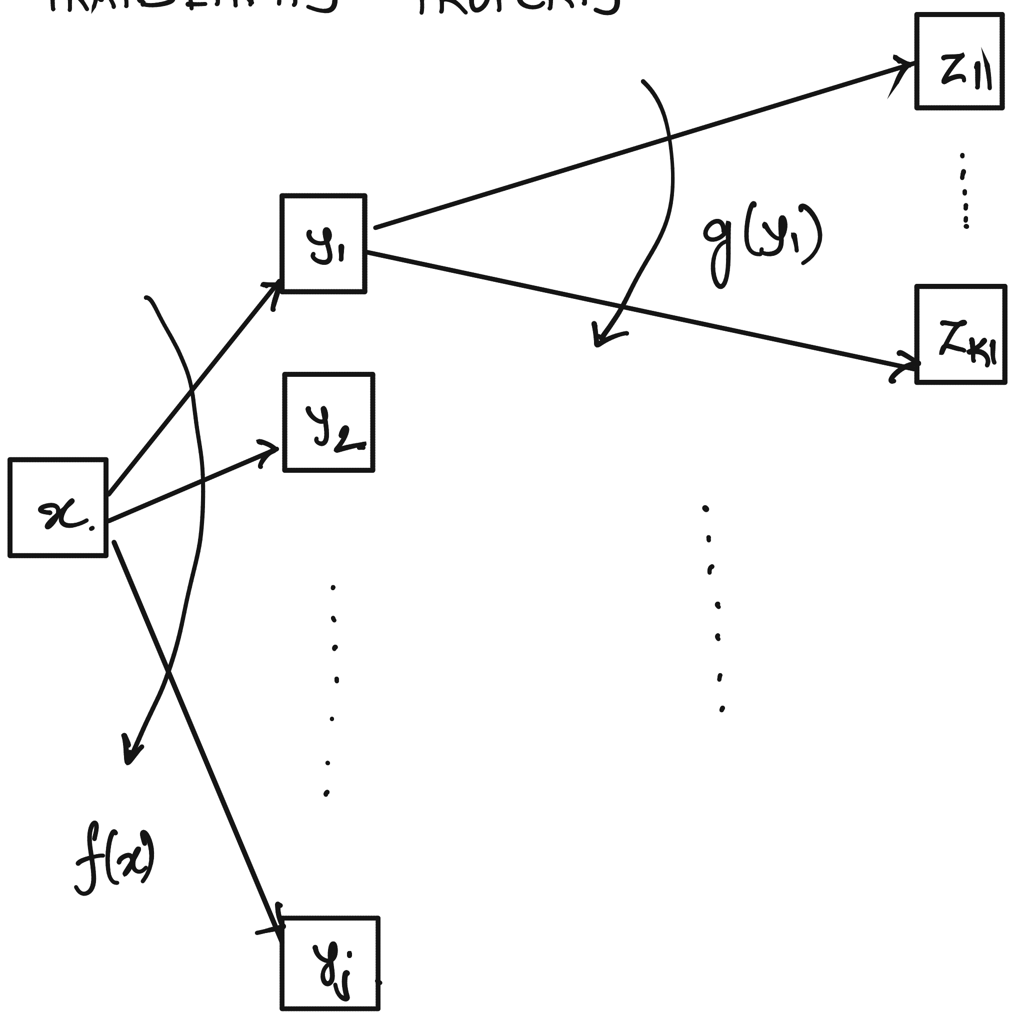
LEMMA: IF $X \le_p Y$ & $Y \le_p Z$, THEN $X \le_p Z$.

TRANSITIVITY PROPERTY



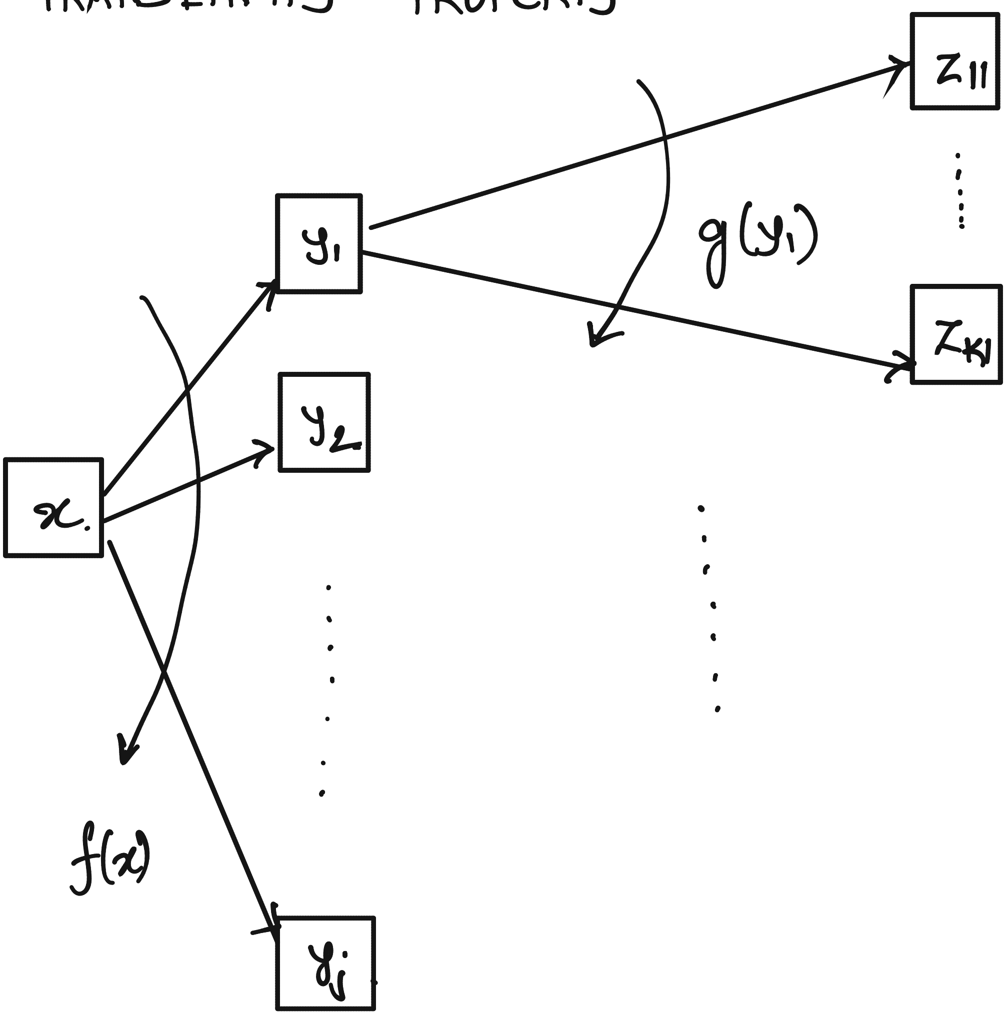
LEMMA: IF $X \leq_p Y$ & $Y \leq_p Z$, THEN $X \leq_p Z$.

TRANSITIVITY PROPERTY



LEMMA : IF $X \le_p Y$ & $Y \le_p Z$, THEN $X \le_p Z$.

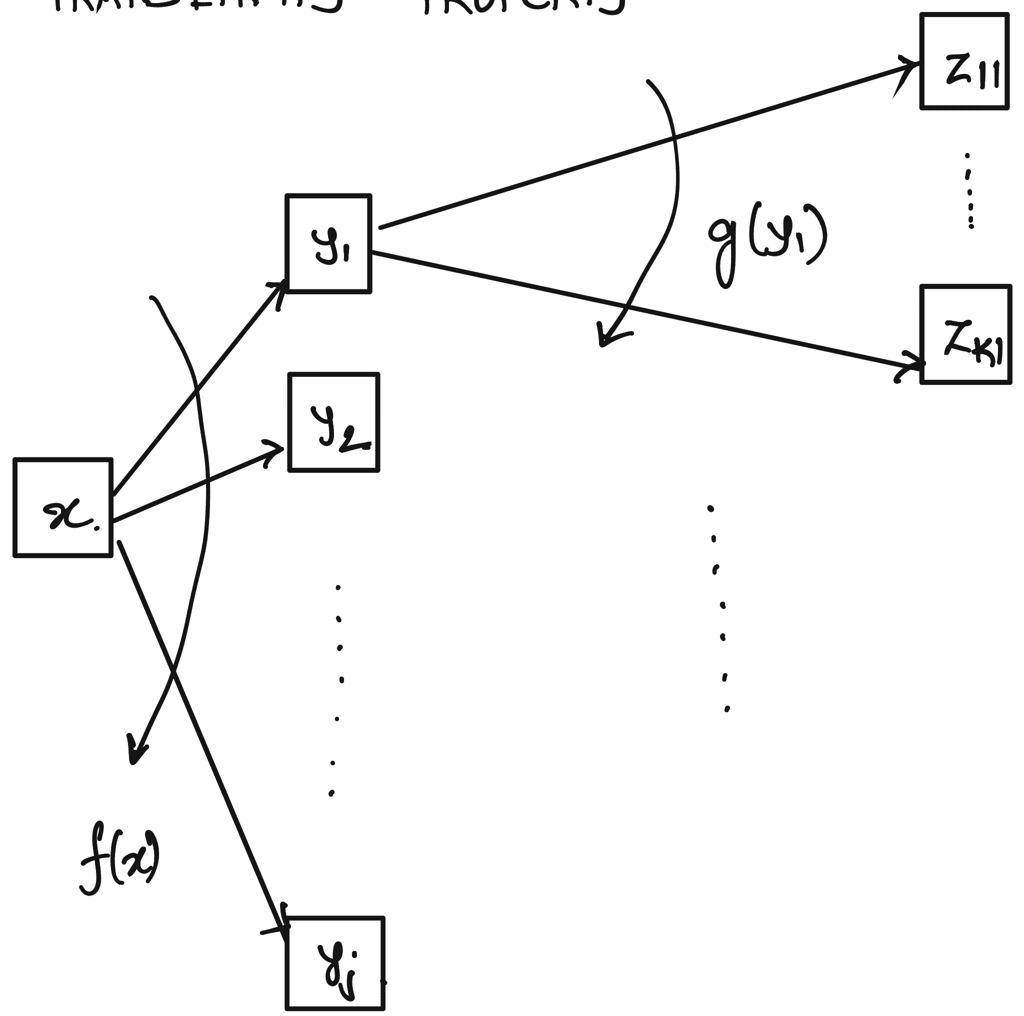
TRANSITIVITY PROPERTY



AN INSTANCE x CAN BE SOLVED IN

LEMMA: IF $X \leq_p Y$ & $Y \leq_p Z$, THEN $X \leq_p Z$.

TRANSITIVITY PROPERTY



AN INSTANCE x CAN BE SOLVED IN

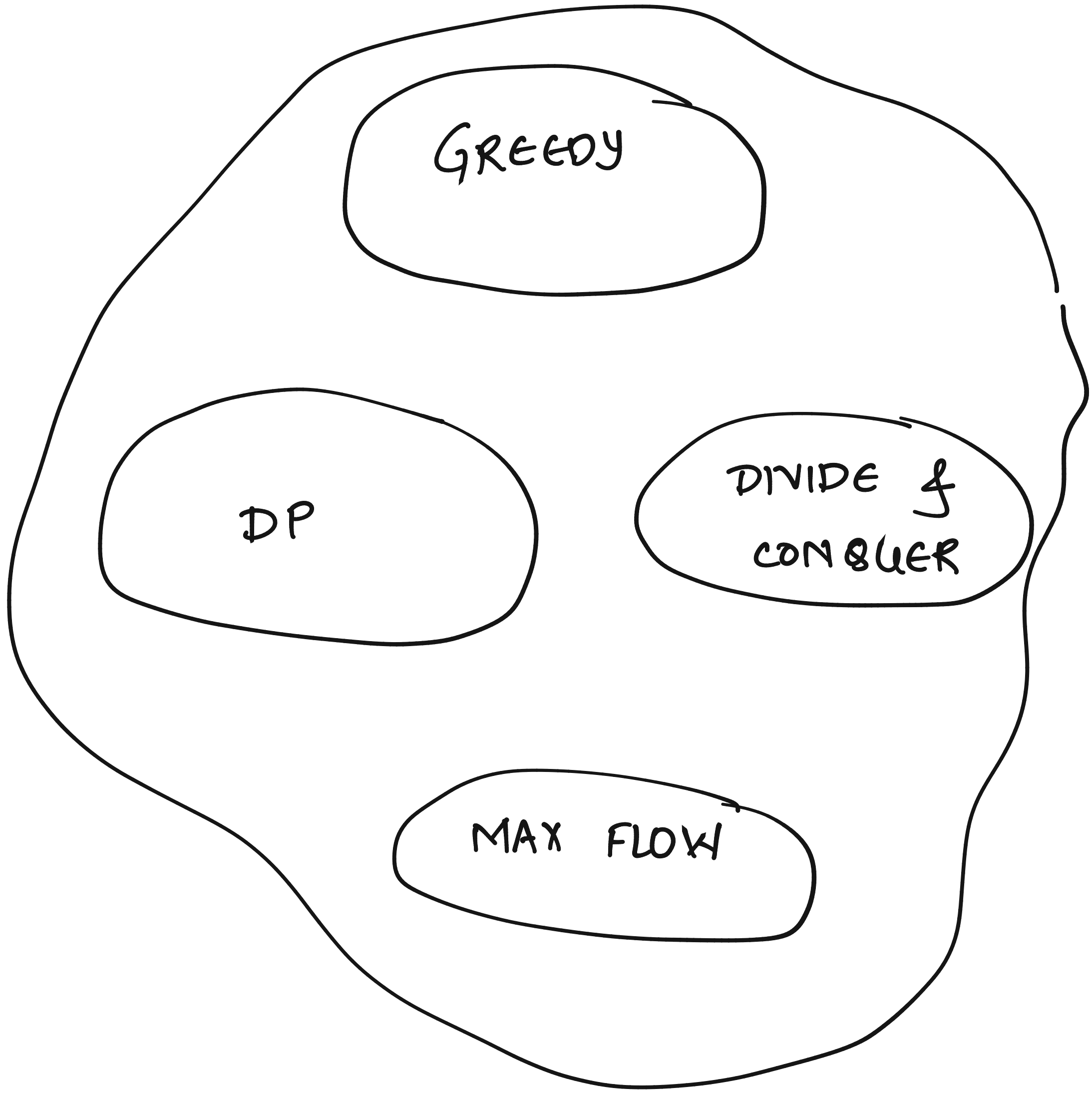
$$f(x) + \sum_{i=1}^{j} g(y_i) + \sum_{i=1}^{j} \sum_{l=1}^{k} \text{RUNNING TIME of INSTANCE } z_{il}.$$

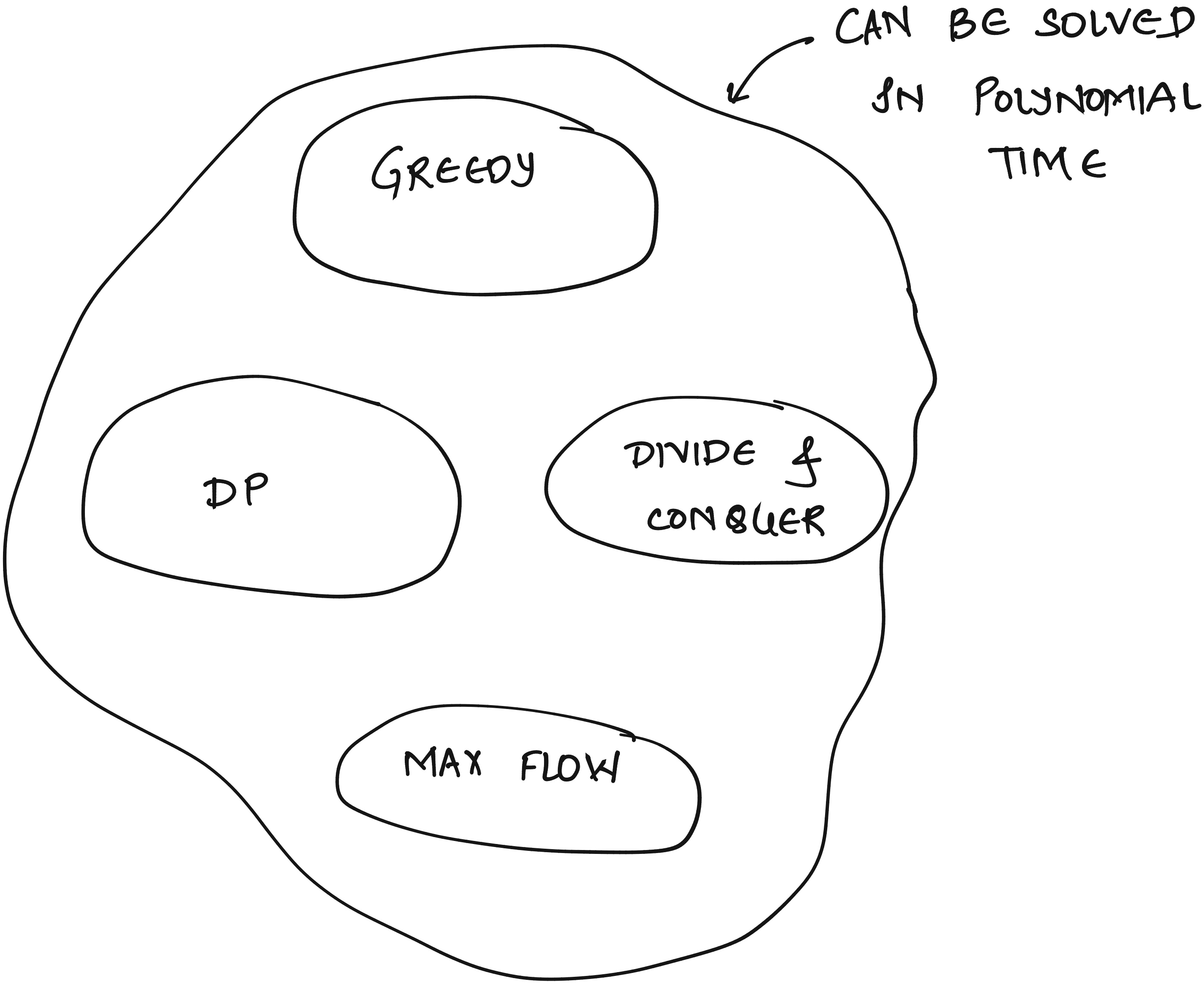
LEMMA: IF $X \le_p Y$, THEN IF Y IS SOLVABLE IN POLYNOMIAL TIME, THEN

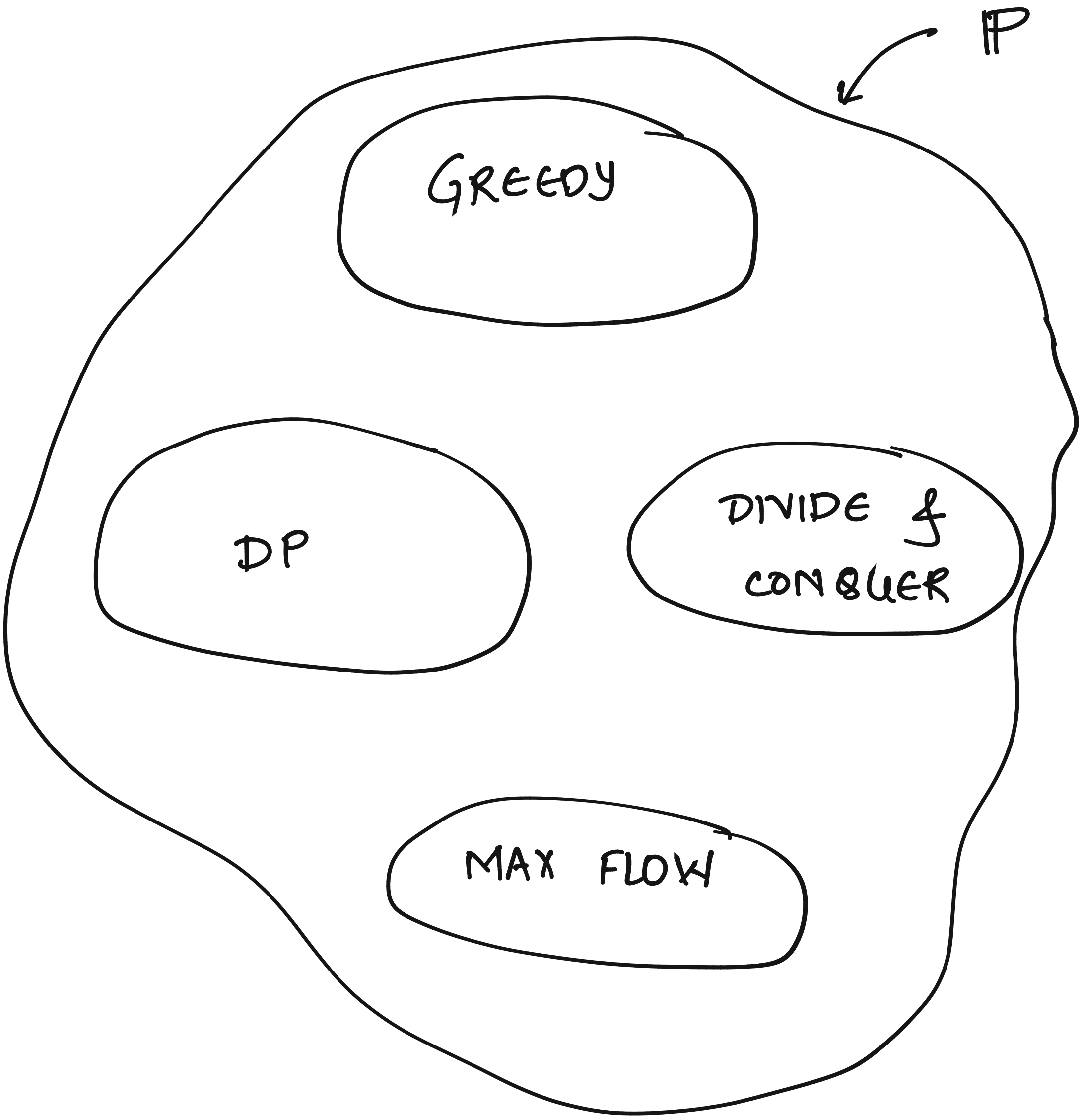
LEMMA: IF $X \leq_p Y$, THEN IF Y IS SOLVABLE IN POLYNOMIAL TIME, THEN X IS ALSO SOLVABLE IN POLYNOMIAL TIME.

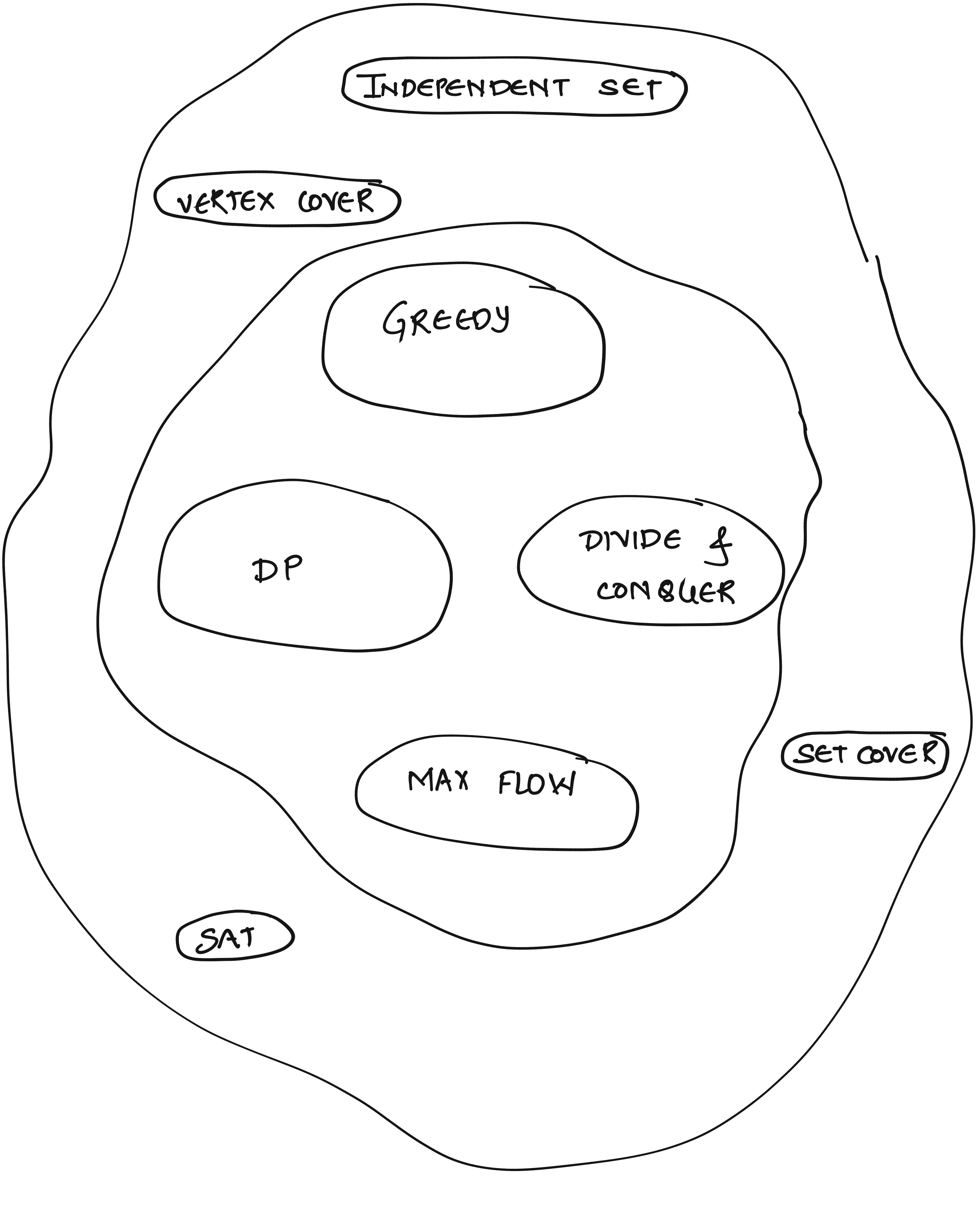
LEMMA: IF $X \leq_p Y$, THEN IF X IS NOT SOLVABLE IN POLYNOMIAL TIME THEN

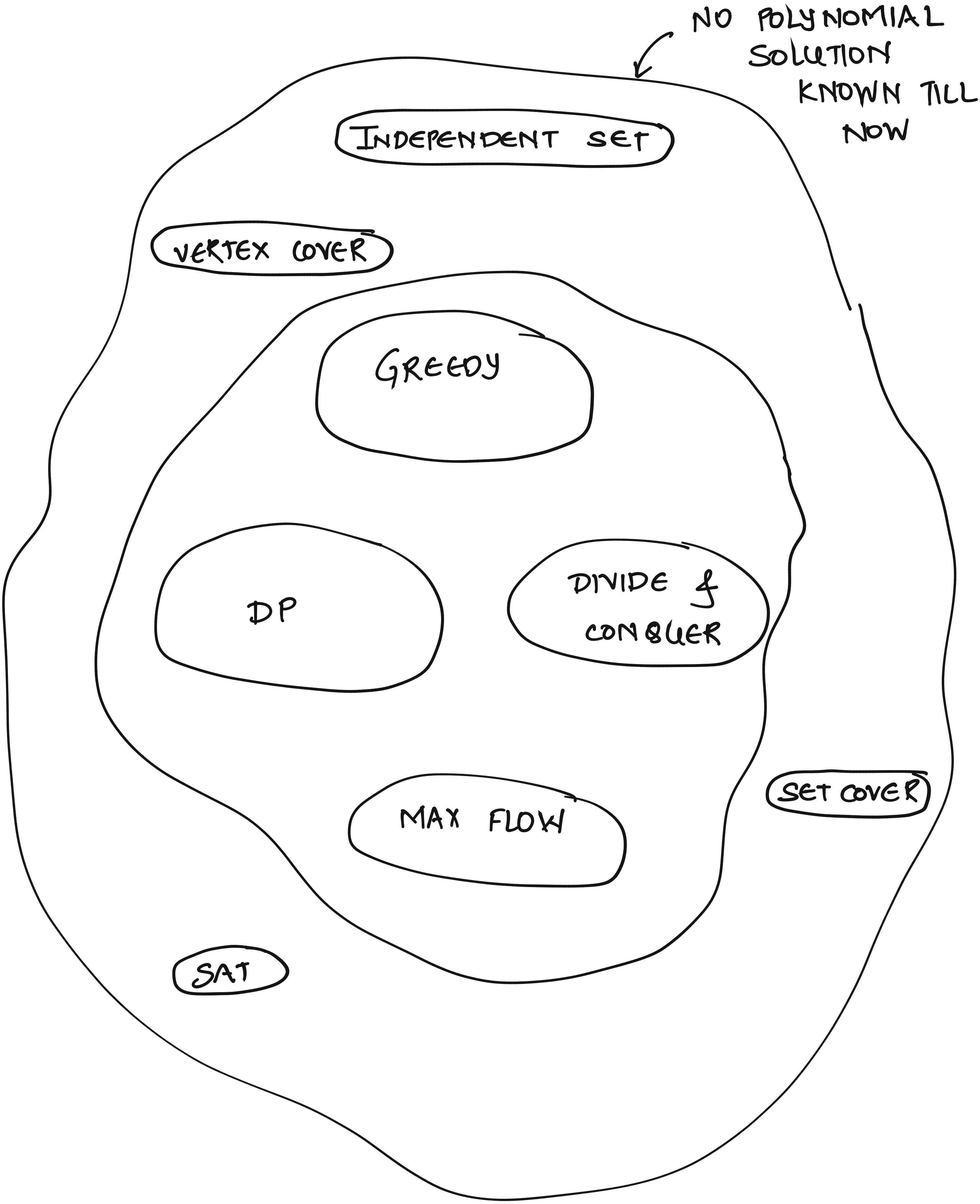
LEMMA: IF $X \leq_p Y$, THEN IF X IS NOT SOLVABLE IN POLYNOMIAL TIME THEN Y IS NOT SOLVABLE IN POLYNOMIAL TIME.

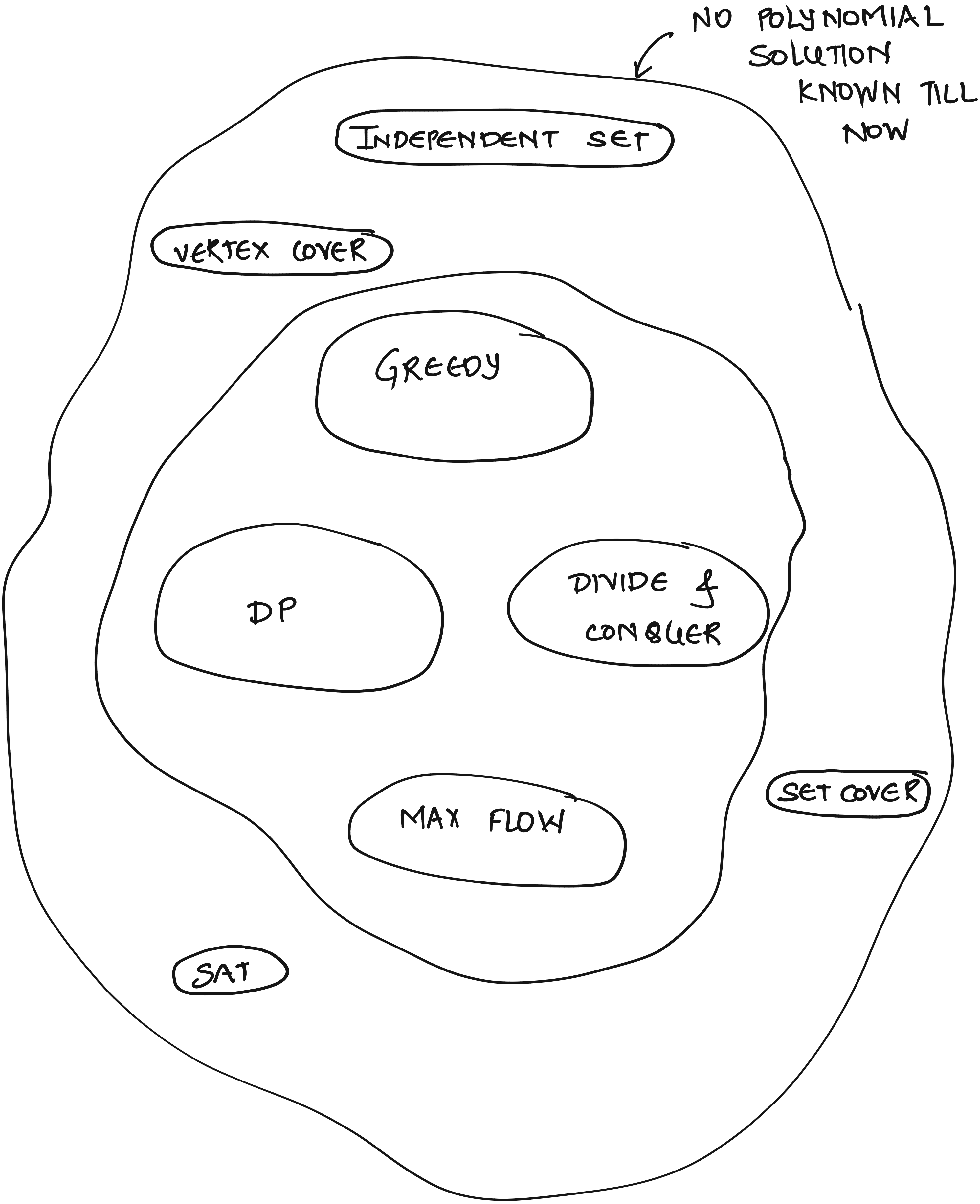

GREEDY
DP
DIVIDE & CONQUER
MAX FLOW


CAN BE SOLVED
IN POLYNOMIAL
TIME
GREEDY
DP
DIVIDE &
CONQUER
MAX FLOW


P
GREEDY
DP
DIVIDE &
CONQUER
MAX FLOW


INDEPENDENT SET
VERTEX COVER
GREEDY
DP
DIVIDE &
CONQUER
MAX FLOW
SET COVER
SAT


NO POLYNOMIAL SOLUTION KNOWN TILL NOW
INDEPENDENT SET
VERTEX COVER
GREEDY
DP
DIVIDE & CONQUER
MAX FLOW
SET COVER
SAT



IS $\leq_p$ VC $\leq_p$ SC $\leq_p$ SAT $\leq_p$ IS

$IS \leq_p VC \leq_p SC \leq_p SAT \leq_p IS$

OBSERVATION

→ A CLASS OF PROBLEMS

→ IF WE CAN SOLVE ANY ONE OF THEM IN POLYNOMIAL TIME, ALL OTHERS CAN BE SOLVED IN POLYNOMIAL TIME.

$IS \leq_p VC \leq_p SC \leq_p SAT \leq_p IS$

<u>OBSERVATION</u>

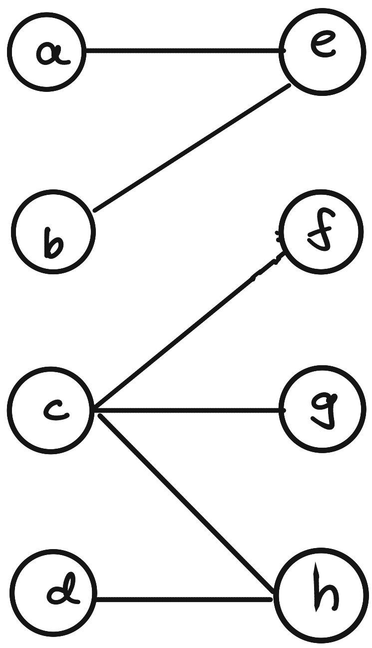
→ A CLASS OF PROBLEMS

→ IF WE CAN SOLVE ANY ONE OF THEM IN POLYNOMIAL TIME, ALL OTHERS CAN BE SOLVED IN POLYNOMIAL TIME.

→ NO POLYNOMIAL TIME IS KNOWN FOR ANY ONE OF THEM.

IS : GIVEN A GRAPH G, AN INDEPENDENT SET IS A VERTEX SET SUCH THAT NO TWO VERTICES IN THE SET ARE ADJACENT TO EACH OTHER

PROBLEM : FIND AN INDEPENDENT SET OF MAXIMUM SIZE



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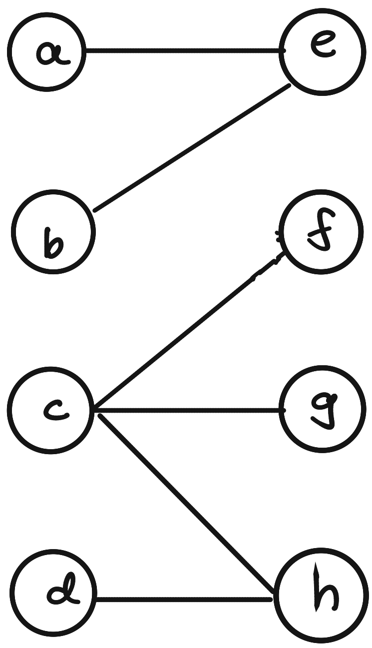

$$MIS = \{ f, g, h, a, b \}$$

VC : VERTEX COVER IS A SET OF VERTICES SUCH THAT EACH EDGE IN THE GRAPH HAS ATLEAST ONE ENDPOINT IN THE SET.

<u>PROBLEM</u>: FIND THE MINIMUM VERTEX COVER

VC : VERTEX COVER IS A SET OF VERTICES SUCH THAT EACH EDGE IN THE GRAPH HAS ATLEAST ONE ENDPOINT IN THE SET.

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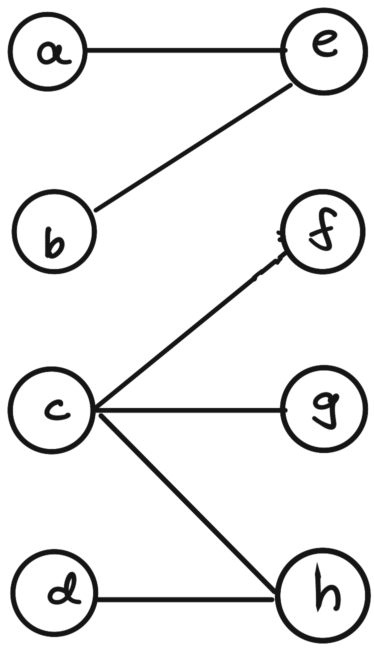

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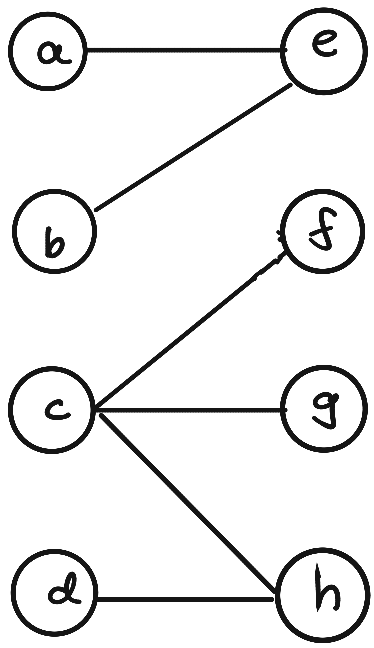
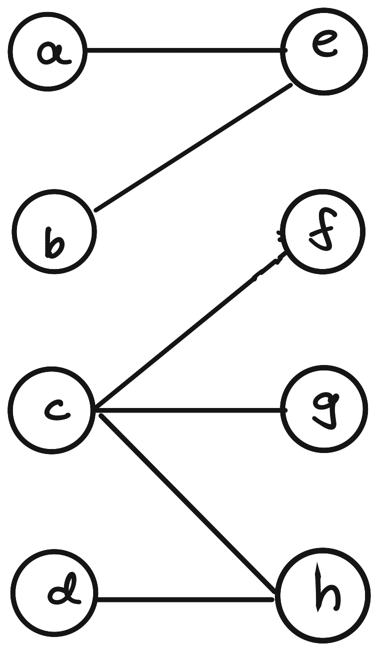


MVC = $\{c, e, d\}$

VC : VERTEX COVER IS A SET OF VERTICES SUCH THAT EACH EDGE IN THE GRAPH HAS ATLEAST ONE ENDPOINT IN THE SET.

PROBLEM: FIND THE MINIMUM VERTEX COVER



MVC = $\{c, e, d\}$

MIS = $\{f, g, h, a, b\}$

VC : VERTEX COVER IS A SET OF VERTICES SUCH THAT EACH EDGE IN THE GRAPH HAS ATLEAST ONE ENDPOINT IN THE SET.

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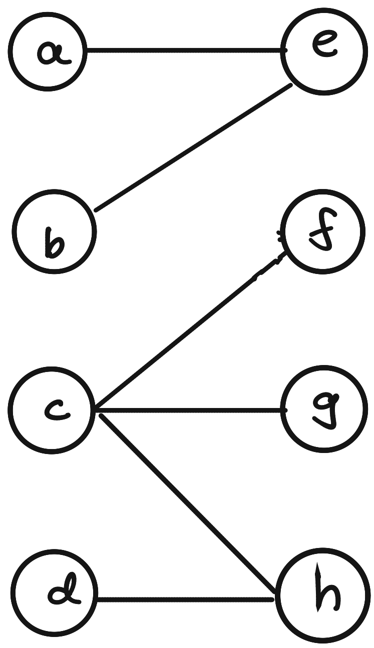

MVC = $\{c, e, d\}$

MIS = V − MVC

MIS = $\{f, g, h, a, b\}$

IS : GIVEN A GRAPH G, AN INDEPENDENT SET IS A VERTEX SET SUCH THAT NO TWO VERTICES IN THE SET ARE ADJACENT TO EACH OTHER

PROBLEM : FIND AN INDEPENDENT SET OF MAXIMUM SIZE

↑

PACKING PROBLEM:

VC : VERTEX COVER IS A SET OF VERTICES SUCH THAT EACH EDGE IN THE GRAPH HAS ATLEAST ONE ENDPOINT IN THE SET.

PROBLEM: FIND THE MINIMUM VERTEX COVER

IS : GIVEN A GRAPH G, AN INDEPENDENT SET IS A VERTEX SET SUCH THAT NO TWO VERTICES IN THE SET ARE ADJACENT TO EACH OTHER

<u>PROBLEM</u> : FIND AN INDEPENDENT SET OF MAXIMUM SIZE

↑

PACKING PROBLEM: PACK AS MANY OBJECTS UNDER SOME CONSTRAINTS.

VC : VERTEX COVER IS A SET OF VERTICES SUCH THAT EACH EDGE IN THE GRAPH HAS ATLEAST ONE ENDPOINT IN THE SET.

<u>PROBLEM</u>: FIND THE MINIMUM VERTEX COVER

IS : GIVEN A GRAPH G, AN INDEPENDENT SET IS A VERTEX SET SUCH THAT NO TWO VERTICES IN THE SET ARE ADJACENT TO EACH OTHER

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↑

PACKING PROBLEM: PACK AS MANY OBJECTS UNDER SOME CONSTRAINTS.

VC : VERTEX COVER IS A SET OF VERTICES SUCH THAT EACH EDGE IN THE GRAPH HAS ATLEAST ONE ENDPOINT IN THE SET.

<u>PROBLEM</u> : FIND THE MINIMUM VERTEX COVER

↑

COVERING PROBLEMS: COVER ALL (OR AS MANY) OBJETS USING MINIMUM NUMBER OF SAME OR OTHER OBJECTS.

IS : GIVEN A GRAPH G, AN INDEPENDENT SET IS A VERTEX SET SUCH THAT NO TWO VERTICES IN THE SET ARE ADJACENT TO EACH OTHER

<u>PROBLEM</u> : IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G ?

↑

PACKING PROBLEM: PACK AS MANY OBJECTS UNDER SOME CONSTRAINTS.

VC : VERTEX COVER IS A SET OF VERTICES SUCH THAT EACH EDGE IN THE GRAPH HAS ATLEAST ONE ENDPOINT IN THE SET.

<u>PROBLEM</u>: IS THERE A VERTEX COVER OF SIZE ATMOST K IN G ?

↑

COVERING PROBLEMS: COVER ALL (OR AS MANY) OBJETS USING MINIMUM NUMBER OF SAME OR OTHER OBJECTS.

PROBLEM : IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G ?

DECISION PROBLEM

PROBLEM : FIND AN INDEPENDENT SET OF MAXIMUM SIZE IN G ?

OPTIMIZATION PROBLEM.

PROBLEM: IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G?

DECISION PROBLEM

PROBLEM: FIND AN INDEPENDENT SET OF MAXIMUM SIZE IN G?

OPTIMIZATION PROBLEM.

CAN YOU SOLVE DP USING OP OF IS?

PROBLEM : IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G ?

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PROBLEM : FIND AN INDEPENDENT SET OF MAXIMUM SIZE IN G ?

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CAN YOU SOLVE DP USING OP OF IS ?

A: YES

PROBLEM : IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G ?

DECISION PROBLEM

PROBLEM : FIND AN INDEPENDENT SET OF MAXIMUM SIZE IN G ?

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CAN YOU SOLVE OP USING DP OF IS ?

PROBLEM : IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G ?

DECISION PROBLEM

PROBLEM : FIND AN INDEPENDENT SET OF MAXIMUM SIZE IN G ?

OPTIMIZATION PROBLEM.

CAN YOU SOLVE DP USING OP OF IS ?

A: YES

CAN YOU SOLVE OP USING DP OF IS ?

A: YES

PROBLEM : IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G ?

DECISION PROBLEM

PROBLEM : FIND AN INDEPENDENT SET OF MAXIMUM SIZE IN G ?

OPTIMIZATION PROBLEM.

CAN YOU SOLVE DP USING OP OF IS ?

A: YES

CAN YOU SOLVE OP USING DP OF IS ?

A: YES

FROM HERE ON WE WILL ASSUME THAT WE ARE SOLVING DECISION VERSION OF ANY PROBLEM

<u>LEMMA</u>: $JS \leq_p VC$.

LEMMA: IS $\leq_p$ VC.

IF S IS A VC, THEN V-S IS AN INDEPENDENT SET.

LEMMA: $IS \le_p VC$.

PROOF:

IF S IS A VC, THEN $V-S$ IS AN INDEPENDENT SET.

$x, y \in V-S$

CAN THERE BE AN EDGE $(x,y) \in G$. ?

LEMMA: IS $\leq_p$ VC.

PROOF:

IF S IS A VC, THEN V-S IS AN INDEPENDENT SET.

$x, y \in V-S$

CAN THERE BE AN EDGE $(x,y) \in G$. ?
NO. OTHERWISE EITHER x OR y SHOULD HAVE BEEN IN VC.

LEMMA: IS $\leq_p$ VC.

PROOF:

IF S IS A VC, THEN V-S IS AN INDEPENDENT SET.

$x, y \in V - S$

CAN THERE BE AN EDGE $(x,y) \in G$. ?
NO. OTHERWISE EITHER x OR y SHOULD HAVE BEEN IN VC.

$\Rightarrow$ V-S IS AN INDEPENDENT SET

LEMMA: IS $\leq_p$ VC.

PROOF:

IF S IS A VC, THEN V-S IS AN INDEPENDENT SET.

$x, y \in V-S$

CAN THERE BE AN EDGE $(x,y) \in G$. ?
NO. OTHERWISE EITHER x OR y SHOULD HAVE BEEN IN VC.

$\Rightarrow$ V-S IS AN INDEPENDENT SET

IS $\rightarrow$ IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G?

LEMMA: IS $\leq_p$ VC.

PROOF:

IF S IS A VC, THEN V-S IS AN INDEPENDENT SET.

$x, y \in V-S$

CAN THERE BE AN EDGE $(x,y) \in G$. ?
NO. OTHERWISE EITHER x OR y SHOULD HAVE BEEN IN VC.

$\Rightarrow$ V-S IS AN INDEPENDENT SET

IS $\rightarrow$ IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G?

IS THERE A VERTEX COVER OF SIZE ATMOST

LEMMA: IS $\leq_p$ VC.

PROOF:

IF S IS A VC, THEN V-S IS AN INDEPENDENT SET.

$x, y \in V-S$

CAN THERE BE AN EDGE $(x,y) \in G$. ?
NO. OTHERWISE EITHER x OR y SHOULD HAVE BEEN IN VC.

$\Rightarrow$ V-S IS AN INDEPENDENT SET

IS $\rightarrow$ IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G?

IS THERE A VERTEX COVER OF SIZE ATMOST $n-k$ IN G?

LEMMA: IS $\leq_p$ VC.

PROOF:

IF S IS A VC, THEN V-S IS AN INDEPENDENT SET.

$x, y \in V-S$

CAN THERE BE AN EDGE $(x,y) \in G$. ?
NO. OTHERWISE EITHER x OR y SHOULD HAVE BEEN IN VC.

$\Rightarrow$ V-S IS AN INDEPENDENT SET

IS $\rightarrow$ IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G?

IS THERE A VERTEX COVER OF SIZE ATMOST $n-k$ IN G?

$\downarrow$ YES $\qquad\qquad$ $\downarrow$ NO

IS: YES

IF S IS AN IS IN G, THEN V-S IS A VC IN G.

IF S IS A IS IN G, THEN V-S IS A VC IN G.

ASSUME FOR CONTRADICTION THAT V-S IS NOT A VC.

IF S IS A IS IN G, THEN V-S IS A VC IN G.

ASSUME FOR CONTRADICTION THAT V-S IS NOT A VC. $\Rightarrow \exists (x, y)$ ST NEITHER x NOR $y \in V-S$

IF S IS A IS IN G, THEN V-S IS A VC IN G.

ASSUME FOR CONTRADICTION THAT V-S IS NOT A VC. $\Rightarrow$ $\exists$ (x,y) ST NEITHER x NOR y $\in$ V-S

$\Rightarrow$ $x, y \in S$

IF S IS A IS IN G, THEN V-S
IS A VC IN G.

ASSUME FOR CONTRADICTION THAT V-S
IS NOT A VC. $\Rightarrow$ $\exists$ (x,y) ST
NEITHER x NOR y $\in$ V-S

$\Rightarrow$ $x, y \in S$

BUT $(x,y) \in G$

$\Rightarrow$ S IS NOT AN INDEPENDENT SET.
$\Rightarrow$ A CONTRADICTION

LEMMA: IS $\leq_p$ VC.

PROOF:

IS $\to$ IS THERE AN INDEPENDENT SET OF SIZE ATLEAST k IN G?

IS THERE A VERTEX COVER OF SIZE ATMOST $n-k$ IN G?

$\downarrow$ YES $\qquad\qquad$ $\downarrow$ NO

IS: YES $\qquad\qquad$ IS: NO

LEMMA: $VC \leq_p IS$.

<u>SET COVER</u>: GIVEN A COLLECTION OF OBJECTS $U = \{u_1, u_2, \dots, u_m\}$ AND A COLLECTION OF SETS $S_1, S_2, \dots, S_n$ SUCH THAT EACH $S_i \subseteq U$, DOES THERE EXISTS ATMOST k SETS WHOSE UNION IS THE SET U.

<u>SET COVER</u>: GIVEN A COLLECTION OF OBJECTS $U = \{u_1, u_2, \dots, u_m\}$ AND A COLLECT OF SETS $S_1, S_2, \dots, S_n$ SUCH THAT EACH $S_i \subseteq U$, DOES THERE EXISTS ATMOST k SETS WHOSE UNION IS THE SET U.

PACKING OR COVERING PROBLEM ?

SET COVER: GIVEN A COLLECTION OF OBJECTS $U = \{u_1, u_2, \dots, u_m\}$ AND A COLLECT OF SETS $S_1, S_2, \dots, S_n$ SUCH THAT EACH $S_i \subseteq U$, DOES THERE EXISTS ATMOST k SETS WHOSE UNION IS THE SET U.

PACKING OR COVERING PROBLEM

LEMMA: VC $\leq_p$ SC

SET COVER: GIVEN A COLLECTION OF OBJECTS $U = \{u_1, u_2, \dots, u_m\}$ AND A COLLECT OF SETS $S_1, S_2, \dots, S_n$ SUCH THAT EACH $S_i \subseteq U$, DOES THERE EXISTS ATMOST k SETS WHOSE UNION IS THE SET U.

PACKING OR COVERING PROBLEM

LEMMA: VC $\leq_p$ SC

$$U = \{e_1, e_2, \dots, e_m\}$$

SET COVER: GIVEN A COLLECTION OF OBJECTS $U = \{u_1, u_2, \dots, u_m\}$ AND A COLLECT OF SETS $S_1, S_2, \dots, S_n$ SUCH THAT EACH $S_i \subseteq \mathcal{U}$, DOES THERE EXISTS ATMOST k SETS WHOSE UNION IS THE SET $\mathcal{U}$.

PACKING OR COVERING PROBLEM

LEMMA: VC $\leq_p$ SC

$$\mathcal{U} = \{e_1, e_2, \dots, e_m\}$$

FOR VERTEX i, MAKE SET

$$S_i = \{ e \mid e \text{ IS ADJACENT TO } i \text{ IN } G \}$$

SET COVER: GIVEN A COLLECTION OF OBJECTS $U = \{u_1, u_2, \ldots, u_m\}$ AND A COLLECT OF SETS $S_1, S_2, \ldots, S_n$ SUCH THAT EACH $S_i \subseteq U$, DOES THERE EXISTS ATMOST k SETS WHOSE UNION IS THE SET U.

PACKING OR COVERING PROBLEM

LEMMA: VC $\leq_p$ SC

$$U = \{e_1, e_2, \ldots, e_m\}$$

FOR VERTEX i, MAKE SET

$$S_i = \{ e \mid e \text{ IS ADJACENT TO } i \text{ IN } G \}$$

IF $\exists$ SC OF SIZE ATMOST K

$\Rightarrow$ $\exists$ A VC OF SIZE ATMOST K

<u>SET COVER</u>: GIVEN A COLLECTION OF OBJECTS $U = \{u_1, u_2, \ldots, u_m\}$ AND A COLLECT OF SETS $S_1, S_2, \ldots, S_n$ SUCH THAT EACH $S_i \subseteq U$, DOES THERE EXISTS ATMOST k SETS WHOSE UNION IS THE SET U.

PACKING OR <u>COVERING PROBLEM</u>

<u>LEMMA</u>: VC $\leq_p$ SC

$$U = \{e_1, e_2, \ldots, e_m\}$$

FOR VERTEX i, MAKE SET

$$S_i = \{ e \mid e \text{ IS ADJACENT TO } i \text{ IN } G \}$$

IF $\exists$ SC OF SIZE ATMOST K

$\Rightarrow$ $\exists$ A VC OF SIZE ATMOST K

IF $\nexists$ SC OF SIZE ATMOST K

$\Rightarrow$

SET COVER: GIVEN A COLLECTION OF OBJECTS $U = \{u_1, u_2, \dots, u_m\}$ AND A COLLECT OF SETS $S_1, S_2, \dots, S_n$ SUCH THAT EACH $S_i \subseteq U$, DOES THERE EXISTS ATMOST k SETS WHOSE UNION IS THE SET U.

PACKING OR COVERING PROBLEM

LEMMA: VC $\leq_p$ SC

$$U = \{e_1, e_2, \dots, e_m\}$$

FOR VERTEX i, MAKE SET

$$S_i = \{ e \mid e \text{ IS ADJACENT TO } i \text{ IN } G \}$$

IF $\exists$ SC OF SIZE ATMOST K

$\Rightarrow$ $\exists$ A VC OF SIZE ATMOST K

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SET PACKING. GIVEN A UNIVERSE $\mathcal{U}$ OF n ELEMENTS & SETS $S_1, S_2, \dots, S_m$ SUBSETS OF $\mathcal{U}$, DOES THERE EXISTS ATLEAST k SETS SUCH THAT NONE OF THEM INTERSET?

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LEMMA: IS $\leq_p$ SP.

SAT:

n BOOLEAN VARIABLES

$x_1, x_2, \ldots, x_n$

SAT:

n BOOLEAN VARIABLES

$x_1, x_2, \dots, x_n$

TERM: x_i OR $\overline{x_i}$

CLAUSE: DISJUNCTION OF DISTINCT

$t_1 \vee t_2 \vee \dots \vee t_\ell$

WHERE $t_i \in \{x_1, x_2, \dots, x_n, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$

SAT:

n BOOLEAN VARIABLES

$x_1, x_2, \ldots, x_n$

TERM: x_i OR $\overline{x_i}$

CLAUSE: DISJUNCTION OF DISTINCT

$t_1 \vee t_2 \vee \ldots \vee t_l$

WHERE $t_i \in \{x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$

$C = x_1 \vee x_2 \vee \overline{x_3}$

SAT:

n BOOLEAN VARIABLES

$x_1, x_2, \ldots, x_n$

TERM : x_i OR $\overline{x_i}$

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$t_1 \vee t_2 \vee \ldots \vee t_\ell$

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$C = x_1 \vee x_2 \vee \overline{x_3}$

C EVALUATES TO 1 IF

SAT:

n BOOLEAN VARIABLES

$x_1, x_2, \dots, x_n$

TERM: x_i OR $\overline{x_i}$

CLAUSE: DISJUNCTION OF DISTINCT

$t_1 \vee t_2 \vee \dots \vee t_\ell$

WHERE $t_i \in \{x_1, x_2, \dots, x_n, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$

$C = x_1 \vee x_2 \vee \overline{x_3}$

C EVALUATES TO 1 IF

$x_1 = 1$

$x_2 = 1$

or $x_3 = 0$

SAT:

n BOOLEAN VARIABLES

$x_1, x_2, \ldots, x_n$

TERM : x_i OR $\overline{x_i}$

CLAUSE: DISJUNCTION OF DISTINCT

$t_1 \vee t_2 \vee \ldots \vee t_\ell$

WHERE $t_i \in \{x_1, x_2, \ldots, x_n, \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$

$C = x_1 \vee x_2 \vee \overline{x_3}$

C EVALUATES TO 1 IF

$$\left.\begin{array}{l} x_1 = 1 \\ x_2 = 1 \\ \text{or } x_3 = 0 \end{array}\right\} \text{assignment.}$$

An assignment satisfies $C_1, C_2, C_3, \ldots, C_K$ clauses if each of them evaluates to 1 wrt the assignment.

$(x_1 \vee \bar{x}_2)$, $(\bar{x}_1 \vee \bar{x}_3)$, $(x_2 \vee \bar{x}_3)$

$(x_1 \vee \bar{x}_2)$, $(\bar{x}_1 \vee \bar{x}_3)$, $(x_2 \vee \bar{x}_3)$

$x_1 = 1$

$x_3 = 0$

$x_2 = 0$ or 1

$(x_1 \vee \bar{x}_2)$, $(\bar{x}_1 \vee \bar{x}_3)$, $(x_2 \vee \bar{x}_3)$

$x_1 = 1$

$x_3 = 0$

$x_2 = 0$ or 1

ASSIGNMENT EVALUATES

$$(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_2 \vee \bar{x}_3)$$

TO 1.

$(x_1 \vee \bar{x}_2)$, $(\bar{x}_1 \vee \bar{x}_3)$, $(x_2 \vee \bar{x}_3)$

$x_1 = 1$

$x_3 = 0$

$x_2 = 0$ or 1

ASSIGNMENT EVALUATES

$$(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_2 \vee \bar{x}_3)$$

TO 1.

<u>SAT</u>: IS THERE AN ASSIGNMENT THAT SATISFIES

$$C_1 \wedge C_2 \wedge C_3 \wedge \dots \wedge C_k$$

WHERE EACH C_i IS A CLAUSE.

$(x_1 \vee \bar{x}_2)$, $(\bar{x}_1 \vee \bar{x}_3)$, $(x_2 \vee \bar{x}_3)$

$x_1 = 1$

$x_3 = 0$

$x_2 = 0$ or 1

ASSIGNMENT EVALUATES

$$(x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee \bar{x}_3) \wedge (x_2 \vee \bar{x}_3)$$

TO 1.

3-SAT: IS THERE AN ASSIGNMENT THAT SATISFIES

$$C_1 \wedge C_2 \wedge C_3 \wedge \dots \wedge C_k$$

WHERE EACH C_i IS A CLAUSE OF LENGTH 3.

LEMMA: 3-SAT $\leq_p$ INDEPENDENT SET

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$$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \ldots \ldots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$$

MAKE A GRAPH OUT OF CLAUSES.

LEMMA : 3-SAT $\leq_p$ INDEPENDENT SET

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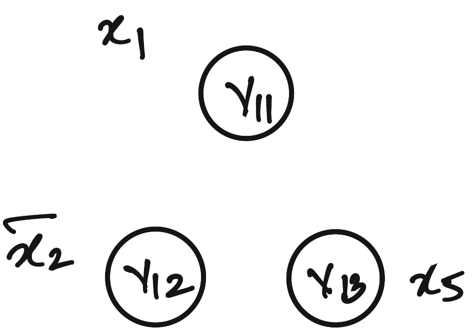
$$\dots\dots \wedge (\bar{x}_3 \vee x_{15} \vee x_2)$$

MAKE A GRAPH OUT OF CLAUSES.



LEMMA: 3-SAT $\leq_p$ INDEPENDENT SET

$$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \ldots$$
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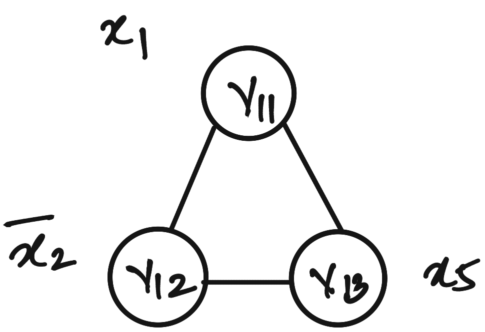
MAKE A GRAPH OUT OF CLAUSES.



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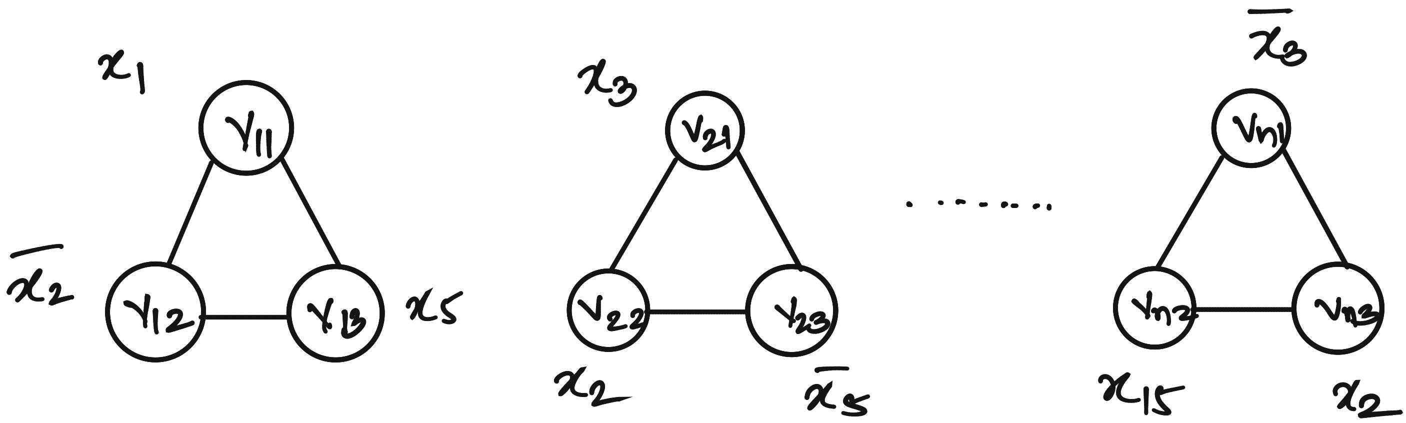

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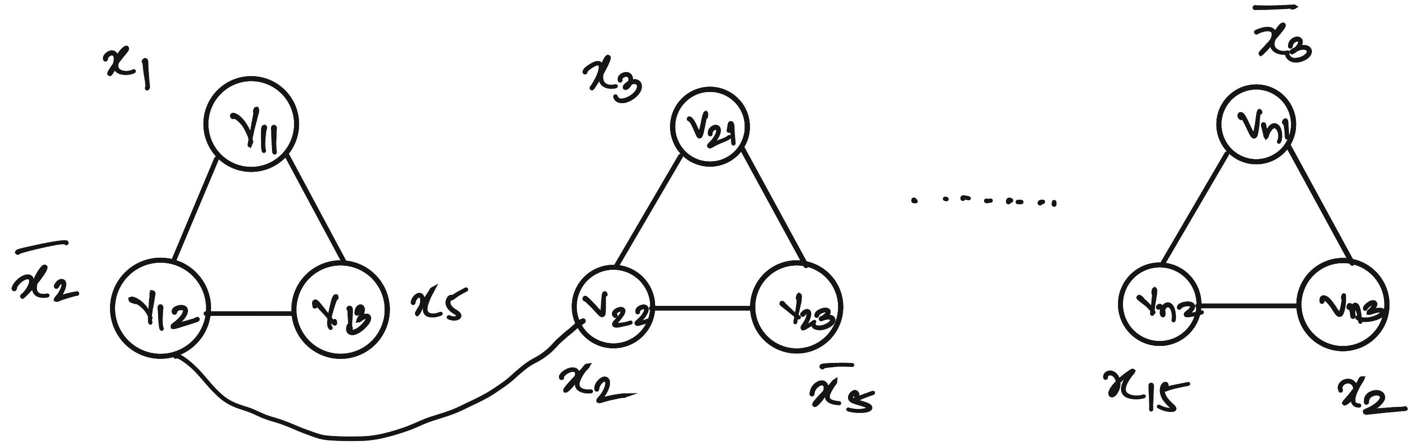
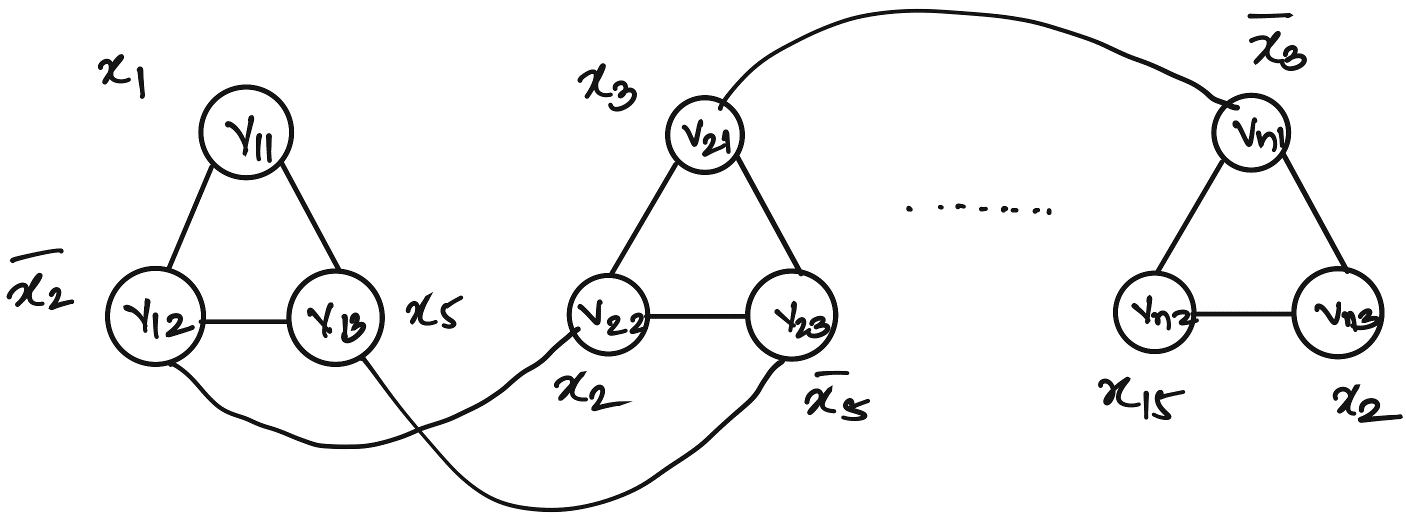


LEMMA: 3-SAT $\leq_p$ INDEPENDENT SET

$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \dots$

$\dots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$

MAKE A GRAPH OUT OF CLAUSES.



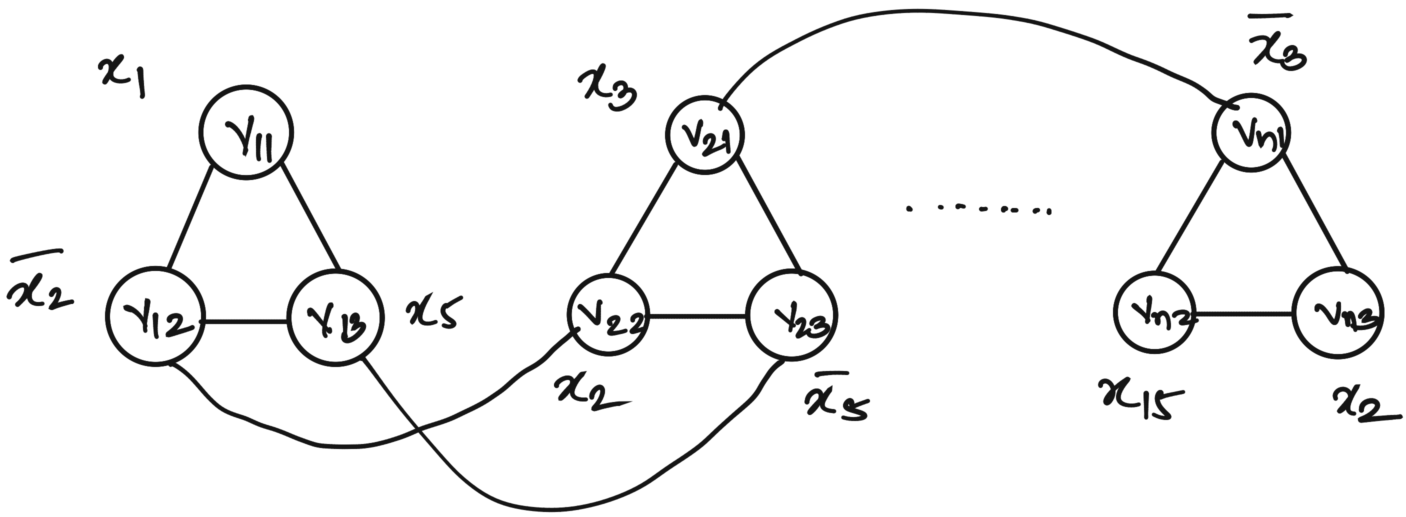
GIVE THIS INPUT TO INDEPENDENT SET BLACKBOX ALGORITHM

LEMMA : 3-SAT $\leq_p$ INDEPENDENT SET

$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \ldots$

$\ldots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$

MAKE A GRAPH OUT OF CLAUSES.



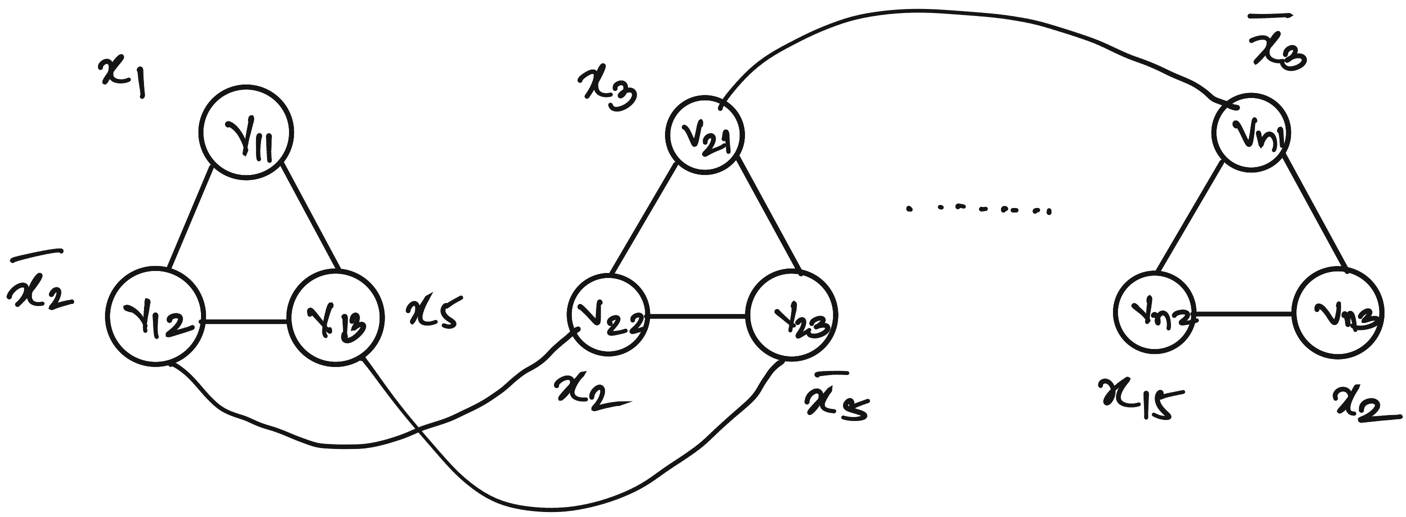
GIVE THIS INPUT TO INDEPENDENT SET BLACKBOX ALGORITHM

Q: IS THERE AN INDEPENDENT SET OF SIZE ATLEAST ___ IN G ?

LEMMA: 3-SAT $\leq_p$ INDEPENDENT SET

$$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \ldots \ldots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$$

MAKE A GRAPH OUT OF CLAUSES.



GIVE THIS INPUT TO INDEPENDENT SET BLACKBOX ALGORITHM

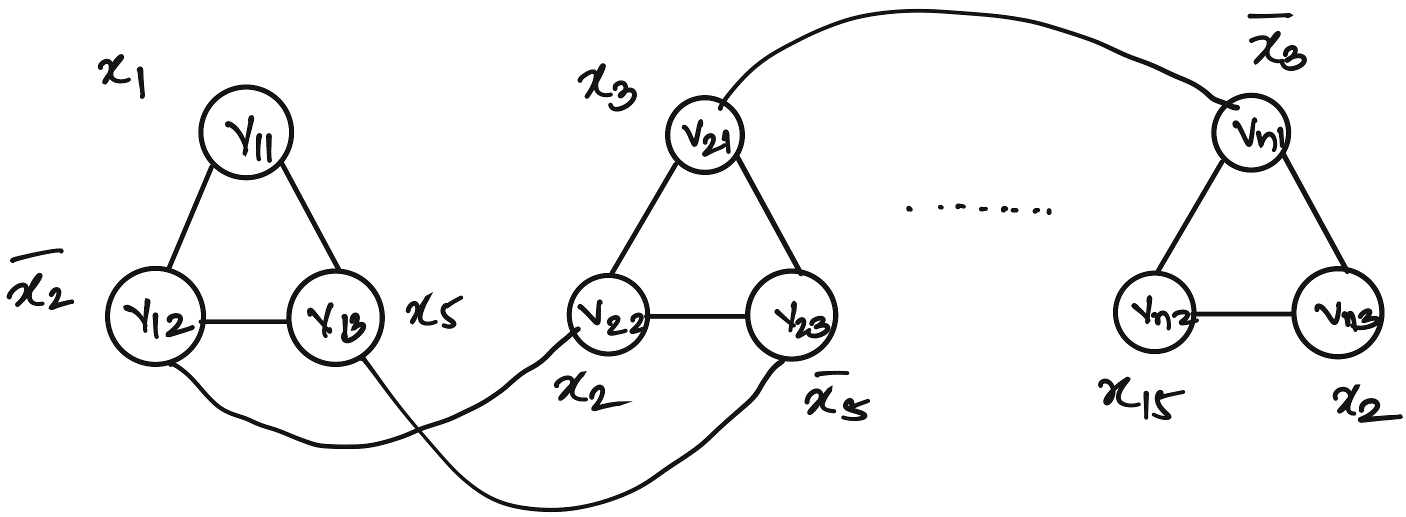
Q: IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G?

LEMMA: 3-SAT $\leq_p$ INDEPENDENT SET

$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \dots$

$\dots\dots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$

MAKE A GRAPH OUT OF CLAUSES.



GIVE THIS INPUT TO INDEPENDENT SET BLACKBOX ALGORITHM

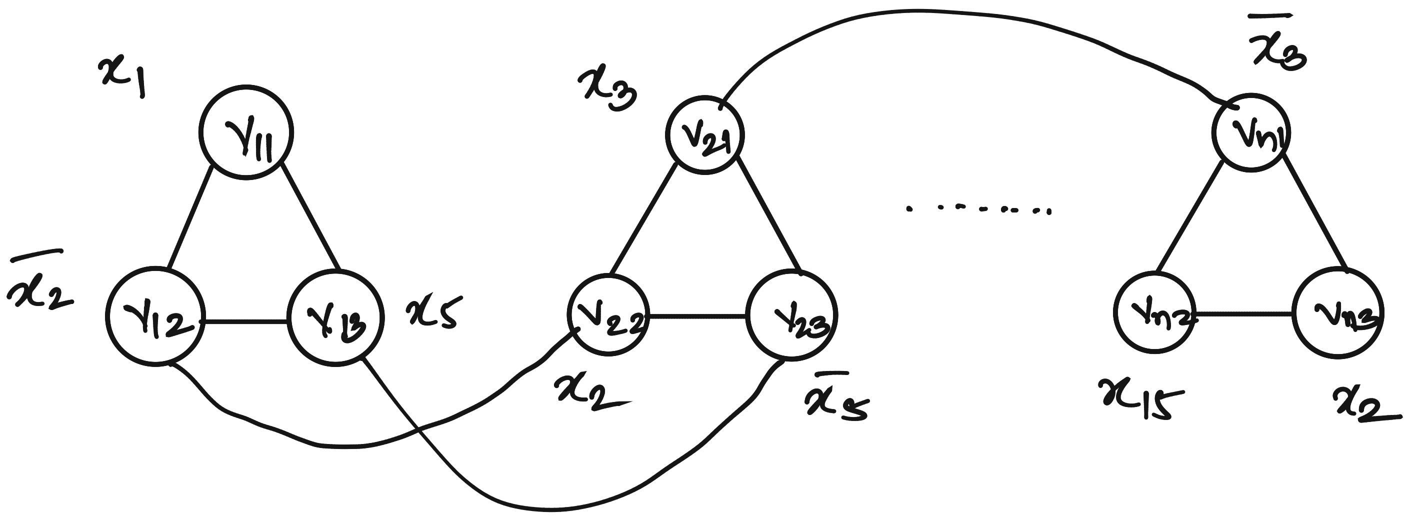
Q: IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G?

A: YES

LEMMA: 3-SAT $\leq_p$ INDEPENDENT SET

$$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \ldots \ldots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$$

MAKE A GRAPH OUT OF CLAUSES.



GIVE THIS INPUT TO INDEPENDENT SET BLACKBOX ALGORITHM

Q: IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G?

A: YES

FOR EACH $v_{ij} \in$ INDEPENDENT SET SET TO CORRESPONDING TERM v_{ij} to 1

NEED TO PROVE THAT

→ EACH CLAUSE EVALUATES TO 1 USING THIS ASSIGNMENT

→ ITS A VALID ASSIGNMENT

NEED TO PROVE THAT

→ EACH CLAUSE EVALUATES TO 1 USING THIS ASSIGNMENT — EASY

→ ITS A VALID ASSIGNMENT

NEED TO PROVE THAT

→ EACH CLAUSE EVALUATES TO 1 USING THIS ASSIGNMENT — EASY

→ ITS A VALID ASSIGNMENT

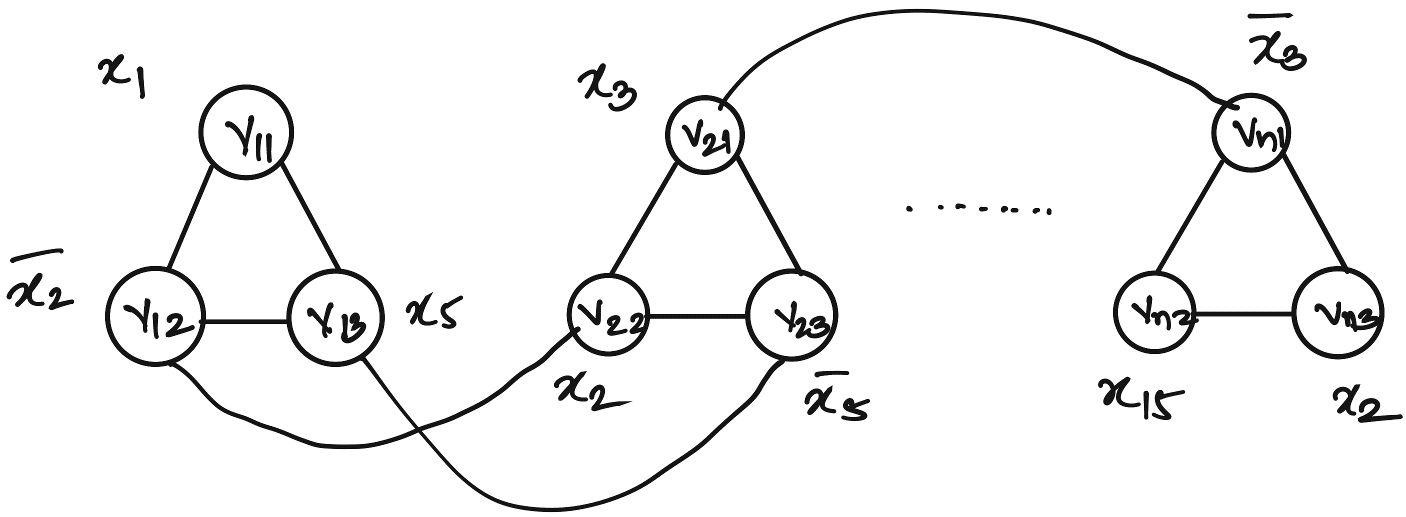
↳ IF $v_{ij} = x_e$ & $v_{i'j'} = \overline{x_e}$

THEN BOTH v_{ij} & $v_{i'j'}$ CANNOT BE SET TO 1.

LEMMA: 3-SAT $\leq_p$ INDEPENDENT SET

$$(x_1 \lor \overline{x_2} \lor x_5) \land (x_3 \lor x_2 \lor \overline{x_5}) \land \ldots$$
$$\ldots \land (\overline{x_3} \lor x_{15} \lor x_2)$$

MAKE A GRAPH OUT OF CLAUSES.



GIVE THIS INPUT TO INDEPENDENT SET BLACKBOX ALGORITHM

Q: IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G?

A: NO

IF $\exists$ AN INDEPENDENT SET OF SIZE K $\rightarrow$ $\exists$ A SATISFYABLE ASSIGNMENT.

IF $\nexists$ AN INDEPENDENT SET OF SIZE K $\rightarrow$ $\nexists$ A SATISFYABLE ASSIGNMENT

IF $\exists$ AN INDEPENDENT SET OF SIZE K $\rightarrow$ $\exists$ A SATISFYABLE ASSIGNMENT.

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IF $\exists$ A SATISFYABLE ASSIGNMENT $\rightarrow$ $\exists$ AN INDEPENDENT SET.

IF $\exists$ AN INDEPENDENT SET OF SIZE K $\rightarrow$ $\exists$ A SATISFYABLE ASSIGNMENT.

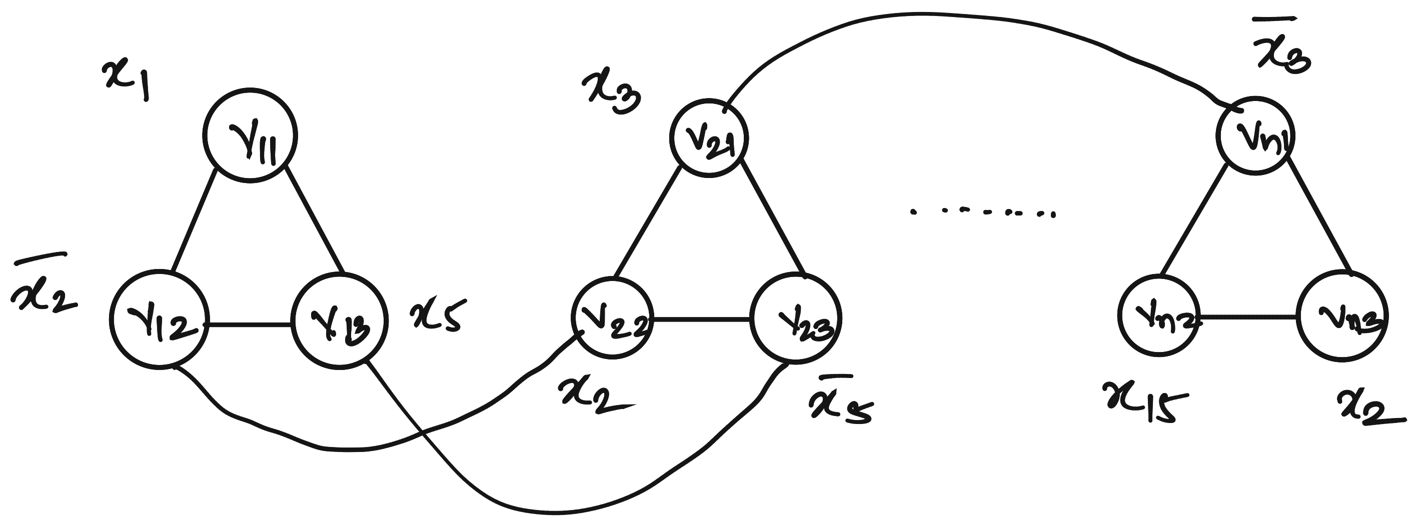
IF $\nexists$ AN INDEPENDENT SET OF SIZE K $\rightarrow$ $\nexists$ A SATISFYABLE ASSIGNMENT

IF $\exists$ A SATISFYABLE ASSIGNMENT $\rightarrow$ $\exists$ AN INDEPENDENT SET ATLEAST K.

$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \dots$

$\dots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$

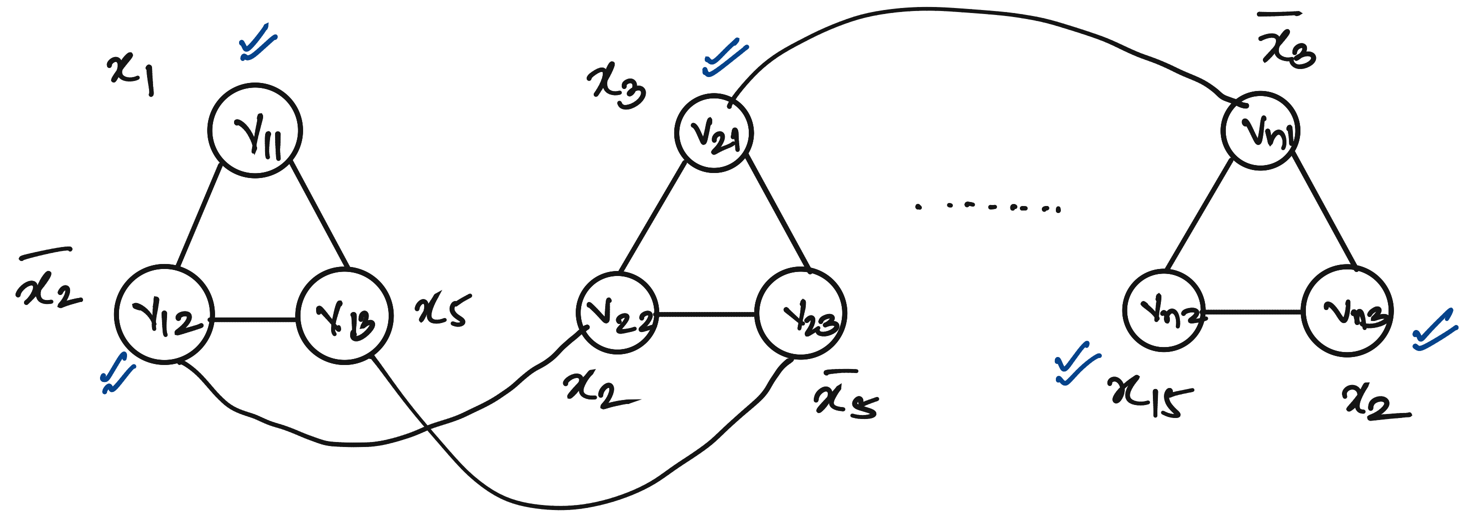
MAKE A GRAPH OUT OF CLAUSES.



LOOK AT ALL TERMS WHICH ARE SET TO 1.

$$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \dots \dots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$$

MAKE A GRAPH OUT OF CLAUSES.

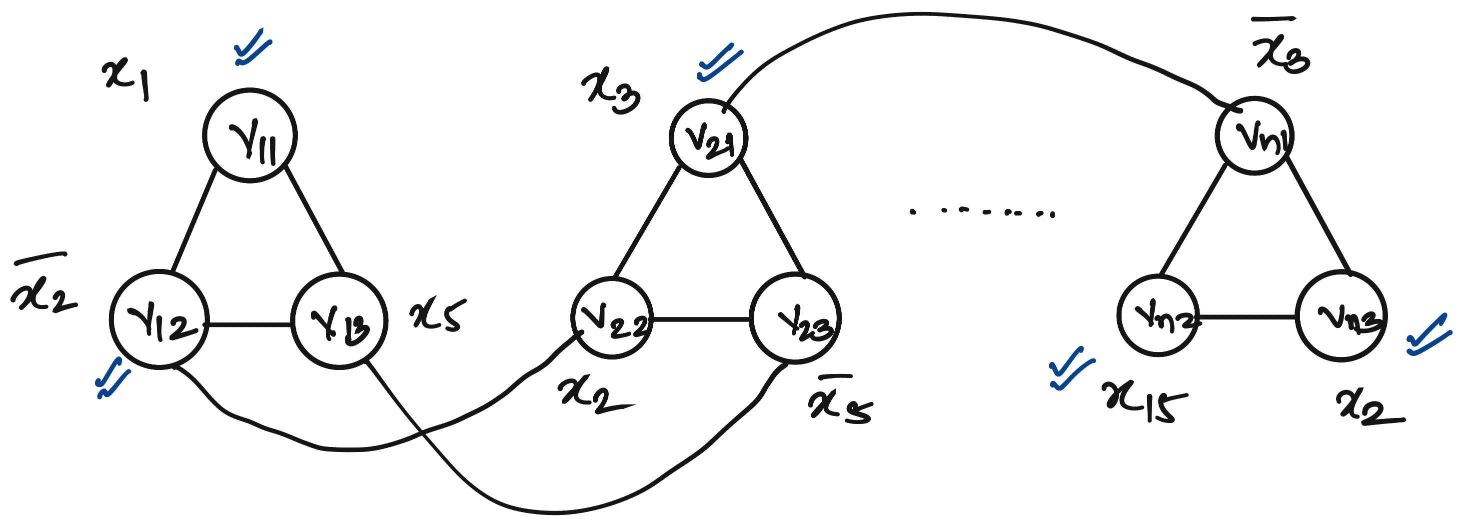


LOOK AT ALL TERMS WHICH ARE SET TO 1.

$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \dots$

$\dots \wedge (\overline{x_8} \vee x_{15} \vee x_2)$

MAKE A GRAPH OUT OF CLAUSES.



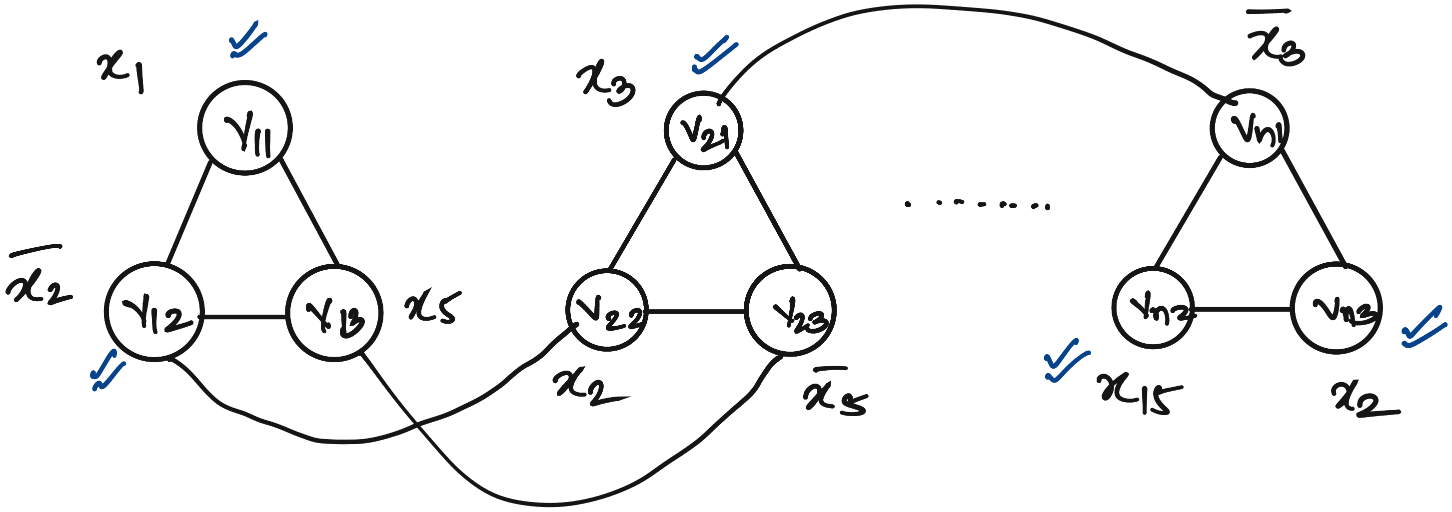
LOOK AT ALL TERMS WHICH ARE SET TO 1.

Q: CAN THERE BE A CLAUSE IN WHICH NO TERM IS SET TO 1?

$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \ldots$

$\ldots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$

MAKE A GRAPH OUT OF CLAUSES.



LOOK AT ALL TERMS WHICH ARE SET TO 1.

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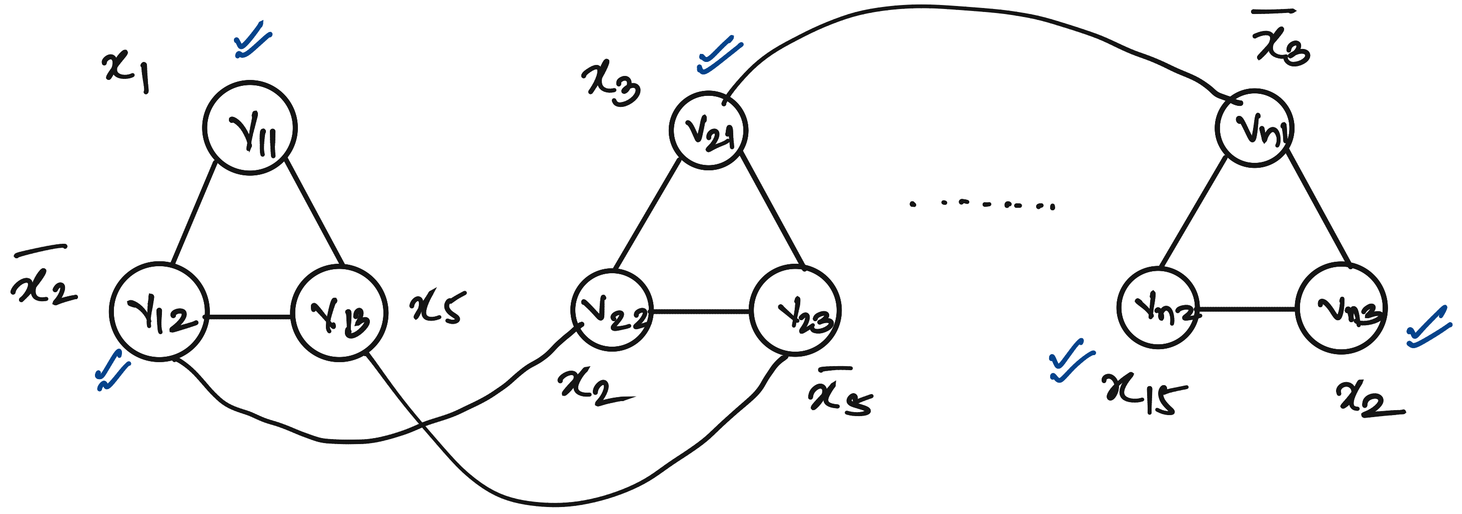
$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \ldots$

$\ldots\ldots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$

MAKE A GRAPH OUT OF CLAUSES.



LOOK AT ALL TERMS WHICH ARE SET TO 1.

Q: CAN THERE BE A CLAUSE IN WHICH NO TERM IS SET TO 1?

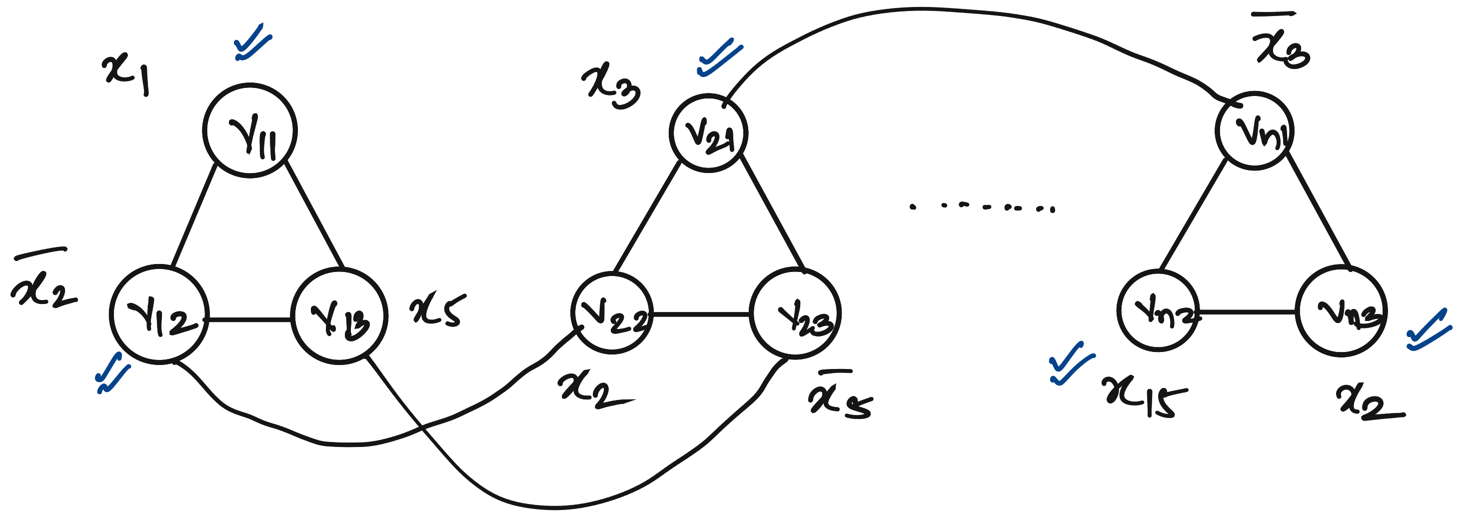
A: NO.

Q: CAN $v_{ij} = x_\ell$ & $v_{i'j'} = \overline{x_\ell}$ BE SET TO 1?

$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \ldots$

$\ldots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$

MAKE A GRAPH OUT OF CLAUSES.



LOOK AT ALL TERMS WHICH ARE SET TO 1.

Q: CAN THERE BE A CLAUSE IN WHICH NO TERM IS SET TO 1?

A: NO.

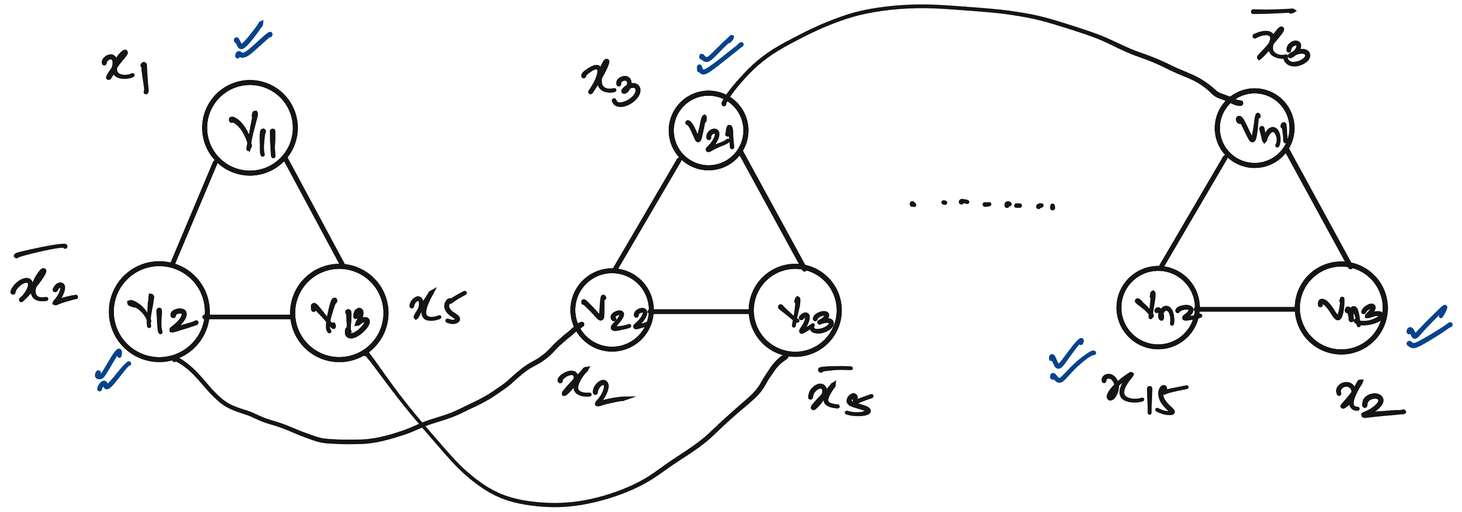
Q: CAN $v_{ij} = x_e$ & $v_{i'j'} = \overline{x_e}$ BE SET TO 1?

A: NO

$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \dots$

$\dots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$

MAKE A GRAPH OUT OF CLAUSES.



LOOK AT ALL TERMS WHICH ARE SET TO 1.

Q: CAN THERE BE A CLAUSE IN WHICH NO TERM IS SET TO 1?

A: NO.

Q: CAN $v_{ij} = x_e$ & $v_{i'j'} = \overline{x_e}$ BE SET TO 1?

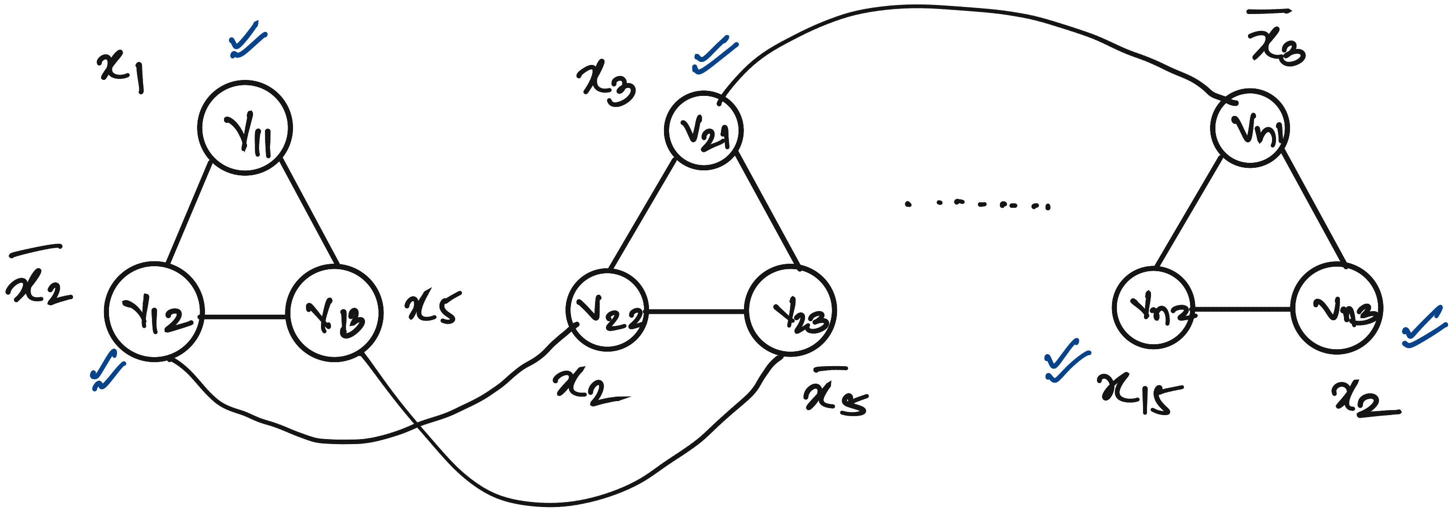
A: NO

$\Rightarrow$ EDGE GOING ACROSS CLAUSES ARE NOT A PROBLEM.

$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \ldots$

$\ldots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$

MAKE A GRAPH OUT OF CLAUSES.

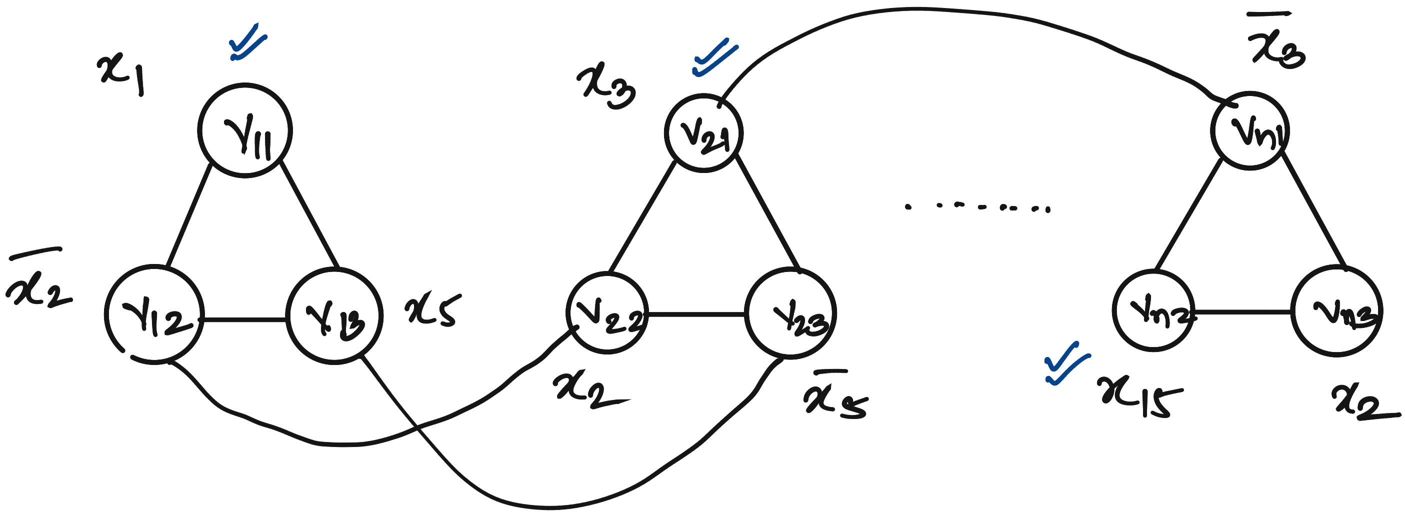


BUT EDGES INSIDE CLAUSES MAY CREATE PROBLEM

$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \ldots$

$\ldots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$

MAKE A GRAPH OUT OF CLAUSES.

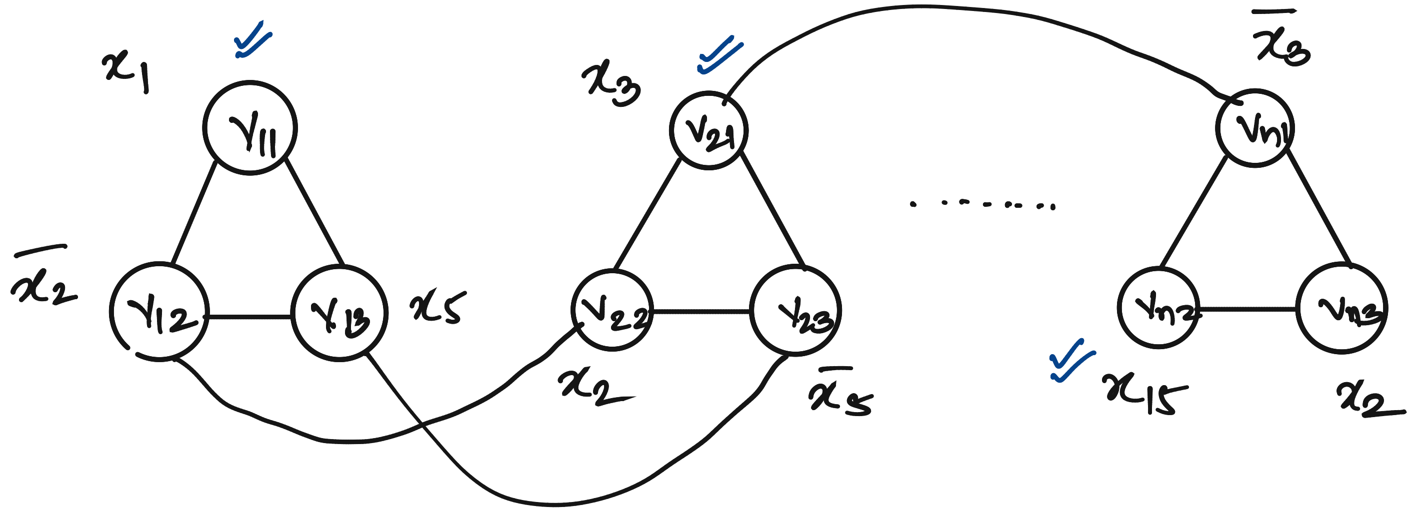


BUT EDGES INSIDE CLAUSES MAY CREATE PROBLEM

$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \dots$

$\dots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$

MAKE A GRAPH OUT OF CLAUSES.



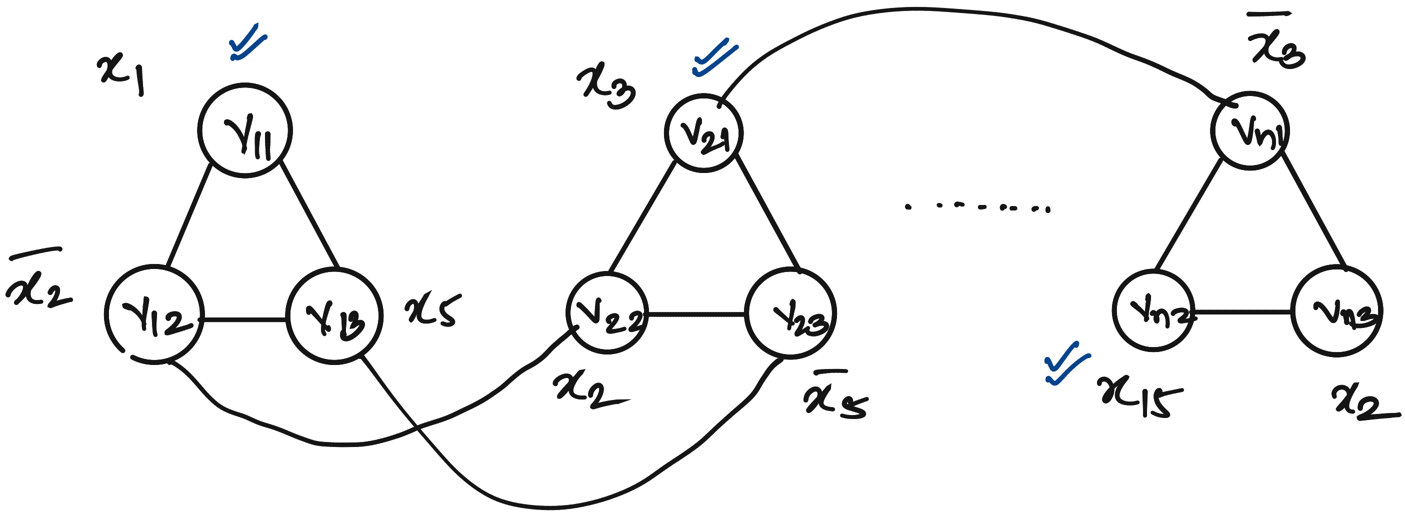
BUT EDGES INSIDE CLAUSES MAY CREATE PROBLEM

IS THIS SET AN INDEPENDENT SET?

$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \ldots$

$\ldots\ldots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$

MAKE A GRAPH OUT OF CLAUSES.



BUT EDGES INSIDE CLAUSES MAY CREATE PROBLEM

IS THIS SET AN INDEPENDENT SET?

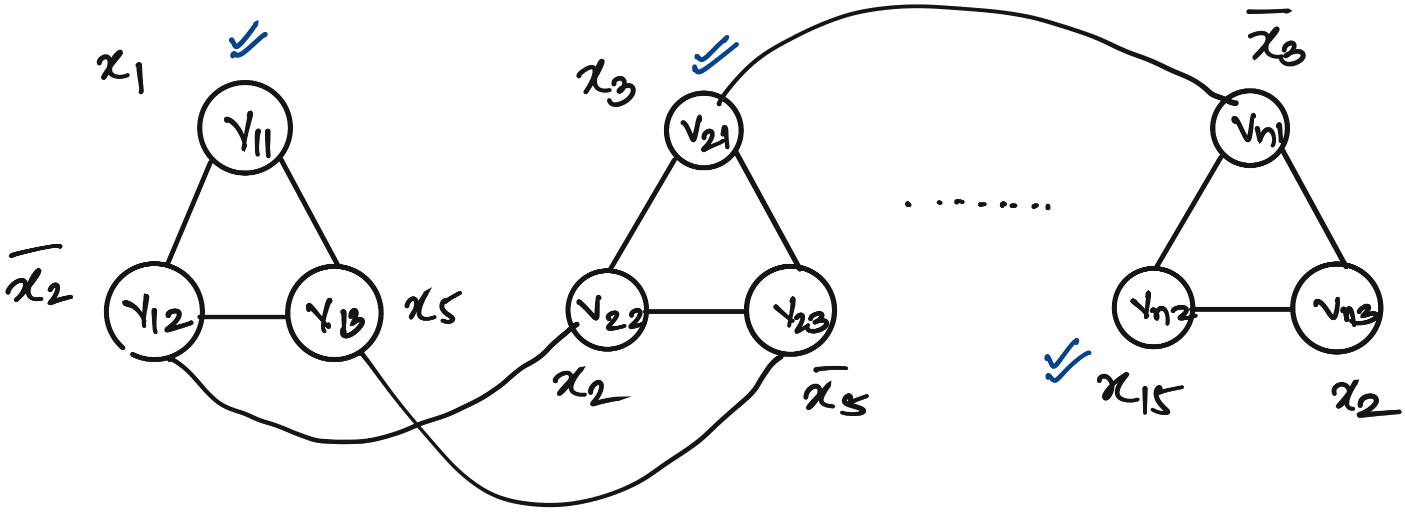
YES

SIZE OF THIS INDEPENDENT SET?

$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \ldots$

$\ldots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$

MAKE A GRAPH OUT OF CLAUSES.



BUT EDGES INSIDE CLAUSES MAY CREATE PROBLEM

IS THIS SET AN INDEPENDENT SET?

YES

SIZE OF THIS INDEPENDENT SET?

K

IF $\exists$ AN INDEPENDENT SET OF SIZE K $\rightarrow$ $\exists$ A SATISFYABLE ASSIGNMENT.

IF $\nexists$ AN INDEPENDENT SET OF SIZE K $\rightarrow$ $\nexists$ A SATISFYABLE ASSIGNMENT

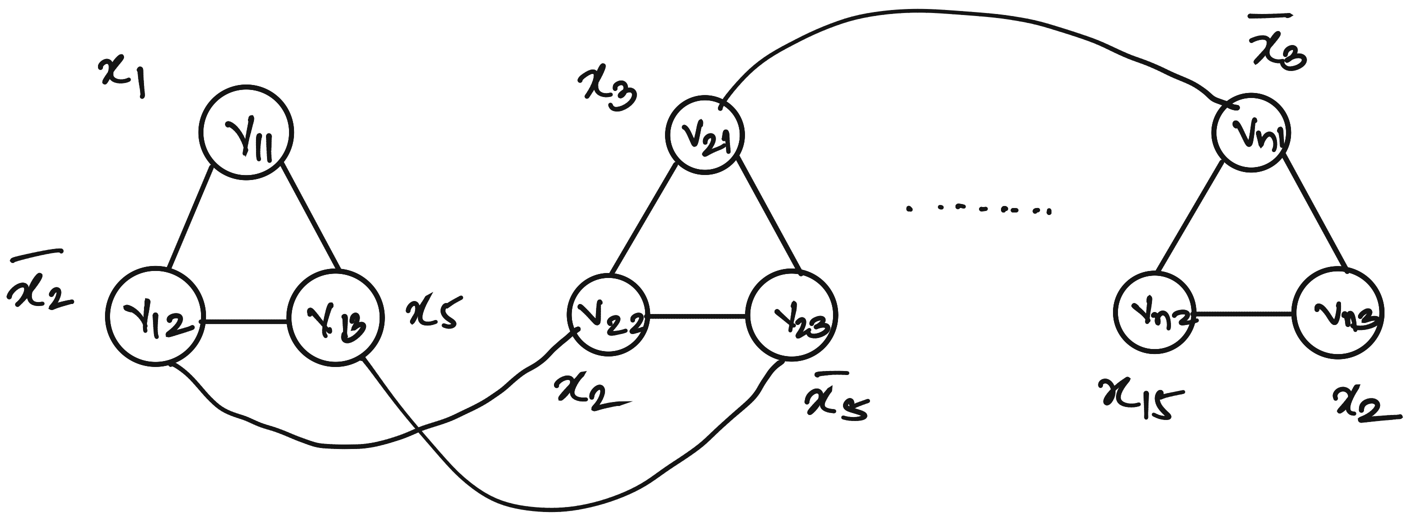
$\downarrow$

IF $\exists$ A SATISFYABLE ASSIGNMENT $\rightarrow$ $\exists$ AN INDEPENDENT SET ATLEAST K.

LEMMA: 3-SAT $\leq_p$ INDEPENDENT SET

$$(x_1 \vee \overline{x_2} \vee x_5) \wedge (x_3 \vee x_2 \vee \overline{x_5}) \wedge \ldots$$
$$\ldots \wedge (\overline{x_3} \vee x_{15} \vee x_2)$$

MAKE A GRAPH OUT OF CLAUSES.



GIVE THIS INPUT TO INDEPENDENT SET BLACKBOX ALGORITHM

Q: IS THERE AN INDEPENDENT SET OF SIZE ATLEAST K IN G?

A: NO

NO SATISFYING ASSIGNMENT.