

General Powers

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Def. Let $z = x + iy$.

$$z^c := e^{c \ln z} \quad (c \in \mathbb{C}, z \neq 0)$$

- $\ln z$ infinitely many-valued implies so is z^c .
- $z^c = e^{c \ln z}$ is the principal value of z^c .
- For $c = 1, 2, 3, \dots$; z^c is single-valued & identical with n^{th} power of z . Similarly for $c = -1, -2, -3, \dots$
- If $c = \frac{1}{n}$, $n = 1, 2, 3, \dots$, then
$$z^c = \sqrt[n]{z} = e^{\frac{1}{n} \ln z}$$
(Unique exponent up to multiples of $\frac{2\pi i}{n}$)
$$\Rightarrow n \text{ distinct values of the } n^{\text{th}} \text{ root.}$$
- If $c = p/q$, again finitely many distinct values
- If c is irrational or complex, z^c is infinitely many-valued

① * What is i^i ?

$$i^i = e^{i \log i} = e^{i(\frac{\pi}{2} \pm 2n\pi i)} = e^{-\frac{\pi}{2} \mp 2n\pi}$$

Thus i^i is always real. Its principle value is $e^{-\pi/2}$.

② $(1+i)^{-1+i} = \exp((-1+i) \ln(1+i))$
$$= \exp((-1+i) (\frac{1}{2} \ln(2) + i(\frac{\pi}{4} \pm 2n\pi)))$$
$$= \exp(-\frac{1}{2} \ln 2 - (\frac{\pi}{4} \pm 2n\pi) + i(-\frac{\pi}{4} \mp 2n\pi + \frac{1}{2} \ln(2)))$$

$$= \frac{1}{\sqrt{2}} e^{\left(-\frac{\pi}{4} + 2n\pi\right)} \cdot \left(\cos\left(-\frac{\pi}{4} + 2n\pi + \frac{1}{2} \ln(2)\right) + i \sin\left(-\frac{\pi}{4} + 2n\pi + \frac{1}{2} \ln(2)\right) \right)$$

Ch. 14.1 - Line integral in the complex plane

- Complex definite integrals are called (complex) line integrals, denoted by $\int_C f(z) dz$, meaning $f(z)$ (the integrand) is integrated over a given curve C in \mathbb{C} known as the path of integration,
- $C: z(t) = x(t) + iy(t) \quad (a \leq t \leq b)$
(Parametric representation)
- C is a smooth curve if its derivative $\dot{z} = \frac{dz}{dt} = \dot{x}(t) + i\dot{y}(t)$ is continuous & non-zero at each point

Definition of the Complex line integral

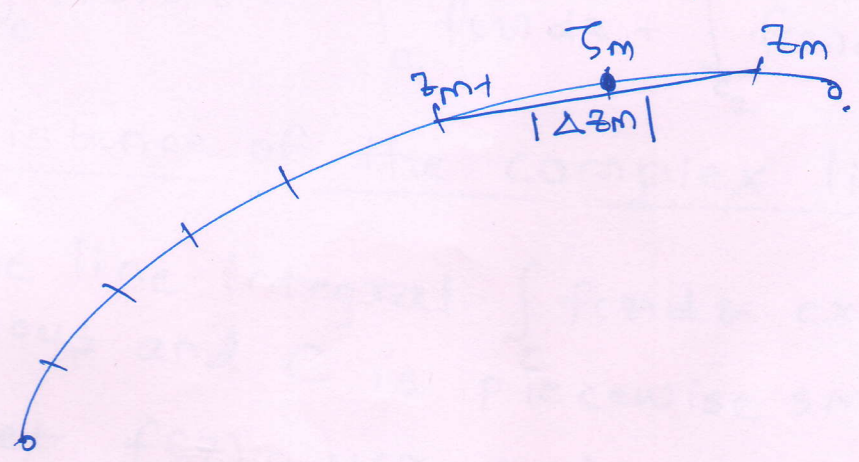
Let C be a smooth curve in the complex plane given by $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, and let f be continuous at each point of C . Partition the interval $a \leq t \leq b$ by points $t_0 (= a), t_1, \dots, t_n (= b)$, where $t_0 \leq t_1 \leq \dots \leq t_n$.

Let the corresponding sub-division of C be $z_0, z_1, \dots, z_n (= Z)$, where $z_j = z(t_j)$.

Choose ζ_1 on part of C between z_0 & z_1 (i.e. $\zeta_1 = z(t)$, $t_0 \leq t \leq t_1$), and similarly, ζ_2 between z_1 and z_2 , etc.

Let $S_n = \sum_{m=1}^n f(\zeta_m) \Delta z_m$, where $\Delta z_m = z_m - z_{m-1}$.

We do a similar thing independently with $n = 2, 3, \dots$, but so that the greatest $|\Delta t_m| = |t_m - t_{m-1}|$ approaches zero as $n \rightarrow \infty$.



- $|\Delta z_m|$ also approaches zero:
 - Arc length of the smooth curve C is a continuous function of t .
 - Since $|t_m - t_{m-1}| \rightarrow 0$, $z_{m-1} \rightarrow z_m$, hence $|\Delta z_m| \rightarrow 0$.

Now $\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{m=1}^n f(\zeta_m) \Delta z_m$ is called the line integral of $f(z)$ over the oriented curve C . This is denoted by $\int_C f(z) dz$ or $\oint_C f(z) dz$.

Assumption: All paths of integration for complex line integrals are assumed to be piecewise smooth, that is, consist of finitely many smooth curves joined end to end.

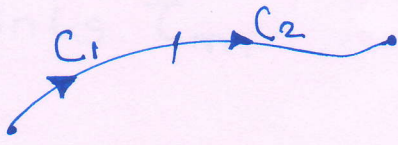
Linearity

$$\int_C (c_1 f_1(z) + c_2 f_2(z)) dz = c_1 \int_C f_1(z) dz + c_2 \int_C f_2(z) dz$$

Sense reversal

$$\int_{z_0}^z f(z) dz = - \int_z^{z_0} f(z) dz$$

Partitioning the path

$$\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz$$


Existence of the complex line integral

The line integral $\int_C f(z) dz$ exists if f is continuous and C is piecewise smooth:

Let $f(z) = u(x, y) + iv(x, y)$.

$$z_m = \xi_m + i\eta_m \quad \& \quad \Delta z_m = \Delta x_m + i\Delta y_m$$

$$\Rightarrow S_n = \sum_{m=1}^n f(z_m) \Delta z_m$$

$$= \sum_{m=1}^n (u + iv)(\Delta x_m + i\Delta y_m),$$

where $u = u(\xi_m, \eta_m)$, $v = v(\xi_m, \eta_m)$

$$\Rightarrow S_n = \sum u \Delta x_m - \sum v \Delta y_m + i (\sum u \Delta y_m + \sum v \Delta x_m)$$

which are real sums

Now f continuous $\Rightarrow u$ and v are continuous
 $n \rightarrow \infty$ implies the greatest Δx_m & $\Delta y_m \rightarrow 0$
Hence we have each sum as real line integral:

$$\lim_{n \rightarrow \infty} S_n = \int_C f(z) dz = \int_C u dx - \int_C v dy + i \left(\int_C u dy + \int_C v dx \right)$$

Since f is continuous & C is piecewise smooth,
each real line integral on the right exists,
 $\Rightarrow \int_C f(z) dz$ exists.

Moreover, its value is independent of the choice
of subdivisions & intermediate points ζ_m .

Evaluating definite integrals

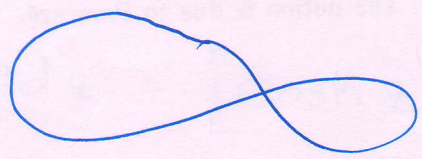
1st method: Indefinite integration and substitution of limits

- Analogue of $\int_a^b f(x) dx = F(b) - F(a)$ ($F'(x) = f(x)$)
- Suitable for analytic functions only.

Defn. A domain D is called simply connected
if every simple closed curve (closed curve
without self-intersections) encloses only points
of D .

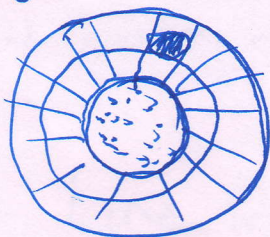


closed curve without self-intersection



closed curve with self-intersection

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• Circular disk is simply connected, but an annulus is not.



• A simply connected domain is one where one can continuously shrink any simple closed curve into a point, while remaining in the domain.