Chapter 6 – Singularities

CLASSIFICATION OF SINGULARITIES

• Isolated singularity – A function f has an isolated singularity at $z=a$ if there is an $R>0$ $\ni$ f is defined and analytic in $B(a;R)-\{a\}$ but not in $B(a;R)$

(We will use the notation $B(a;R)'$ to denoted $B(a;R)-\{a\}$.)

Notation: Let $\mathcal{A}(G)$ denote the space of all analytic functions on G.

• Examples of functions having isolated singularities

(1) $\frac{\sin z}{z}$, $\sin(\frac{1}{z})$, $\frac{e^z}{z-1}$, $\frac{z^2-1}{z-1}$, $\frac{1}{e^z-1}$, $\frac{z^4+1}{z^2+1}$, $\frac{z}{e^z-1}$,

• Example of a function which does not have isolated singularities:

$\log z$.

(1) Removable singularity – Let f have an isolated singularity at a. The point a is called a removable singularity if $\exists$ $g: B(a;R) \to \mathbb{C}$ $\ni$ $g(z)=f(z)$ for $0<|z-a|<R$.

Eg. $\frac{\sin z}{z}$, $\frac{z^2-1}{z-1}$

Thm 6.1 If f has an isolated singularity at a, then the point $z=a$ is a removable singularity iff

$$\lim_{z \to a} (z-a) f(z) = 0.$$

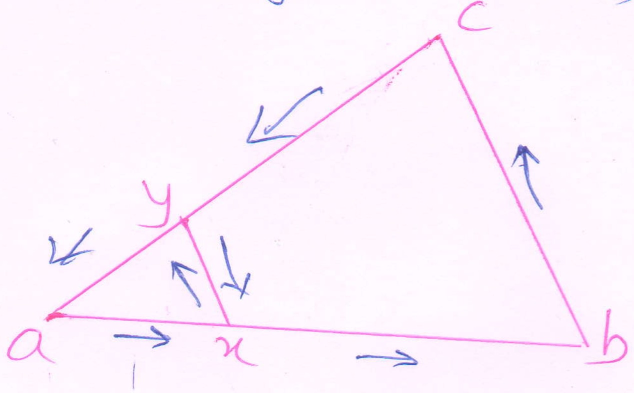
Proof: Suppose f is analytic in $B(a;R)'$. Define

$$g(z) = \begin{cases} (z-a)\, f(z) & \text{for } z \neq a \\ 0 , & \text{for } z = a . \end{cases}$$

Suppose $\lim_{z \to a} (z-a) f(z) = 0$. Then $\lim_{z \to a} g(z) = g(a)$,
So g is continuous on $B(a;R)$.

If we show g is analytic, it follows that 'a' is a removable singularity.

Let T be a triangle in $B(a;R)$ with $\Delta = T$ + its interior.



If $a \notin \Delta$, then $T \approx 0$ in $B(a;R)'$. So $\int_T g = 0$ by Cauchy's thm. (2nd version, since the annulus is open and connected.)

If a is a vertex of $T = [a,b,c,a]$, and $x \in [a,b]$,
$y \in [c,d]$ form $T_1 = [a,x,y,a]$,
polygon $[x,b,c,y,x]$ with P being the then

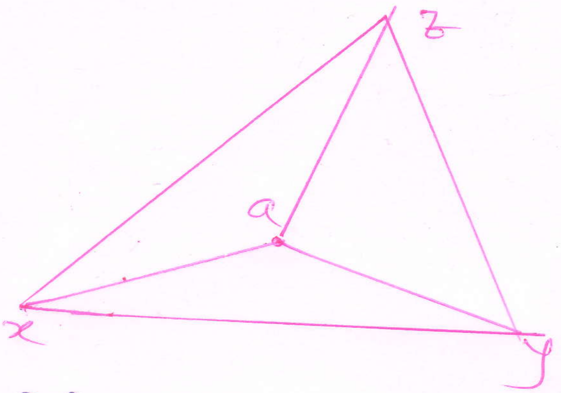
$$\int_T g = \int_{T_1} g + \int_P g = \int_{T_1} g \quad (\because P \approx 0 \text{ in the punctured disk})$$

Since g is continuous and $g(a)=0$, given $\varepsilon > 0$, we can choose x & y s.t. $|g(z)| \leq \frac{\varepsilon}{\ell}$ (where ℓ is the length of T_1) for any z on T_1

$\Rightarrow \left|\int_T g\right| = \left|\int_{T_1} g\right| \leq \frac{\varepsilon}{l} \cdot l(T_1) \leq \frac{\varepsilon}{l} \cdot l = \varepsilon.$

$\Rightarrow \int_T g = 0.$

If $a \in \Delta$, and $T = [x, y, z, x]$, then consider the



Δ's $T_1 = [x, y, a, x]$

$T_2 = [y, z, a, y]$

$T_3 = [z, x, a, z]$,

Then $\int_{T_j} g = 0 \quad 1 \leq j \leq 3.$

$\Rightarrow \int_T g = \int_{T_1} g + \int_{T_2} g + \int_{T_3} g = 0.$

$\Rightarrow$ By Morera's thm, g is analytic.

"$\Rightarrow$" If f has a removable singularity at $z = a$, then $\exists\, h \in \mathcal{A}(B(a;R)) \ni f(z) = h(z)$ on $B(a;R)'$.

Then $\lim_{z \to a} (z-a) f(z) = \lim_{z \to a} (z-a) h(z) = 0.$

$\boxtimes$

Defn. If $z=a$ is an isolated singularity of f, then a is a pole of f if $\lim_{z\to a} |f(z)| = \infty$, that is, for any $M>0$, $\exists\, \varepsilon>0 \ni |f(z)| \geq M$ whenever $0<|z-a|<\varepsilon$.

Defn. An isolated singularity that is neither a pole nor a removable singularity, is called an essential singularity.

Example of a pole: Let $f(z) = \dfrac{1}{(z-a)^m}$, $m \in \mathbb{N}$.
Then f has a pole (of order m) at $z=a$.

Example of an essential singularity:

- $\lim_{z\to 0} \exp\left(\frac{1}{z}\right) \neq 0 \implies z=0$ is not a removable singularity of $\exp(1/z)$.
- $\lim_{z\to 0} \left|\exp\left(\frac{1}{z}\right)\right| \neq \infty$ as can be seen by letting $z\to 0$ over negative reals. Hence $z=0$ is not pole of $\exp(1/z)$.

Thus together, we see that $z=0$ is an essential singularity of $\exp(1/z)$.

Thm. 6.2 If G is a region with a in G and if f is analytic on $G\setminus\{a\}$ with a pole at $z=a$ then there is a positive integer m and an analytic function $g: G \to \mathbb{C} \ni f(z) = \dfrac{g(z)}{(z-a)^m}$.

Proof: Suppose f has a pole at $z=a$. Then, $\dfrac{1}{f(z)}$ has a removable singularity at $z=a$ ($\because \lim_{z\to a} |f(z)| = \infty$ implies $\lim_{z\to a} \left|\dfrac{1}{f(z)}\right| = 0 \implies \left|\lim_{z\to a} \dfrac{1}{f(z)}\right| = 0 \implies \lim \dfrac{1}{\ldots} = 0$

$$\lim_{z \to a} (z-a) \cdot \frac{1}{f(z)} = 0).$$

Then $h(z) := \begin{cases} \frac{1}{f(z)}, & z \neq a \\ 0, & z = a \end{cases}$ is analytic in $B(a;R)$ for some $R > 0$.

But $h(a) = 0$ implies $h(z) = (z-a)^m h_1(z)$ for some analytic fn. h_1 s.t. $h_1(a) \neq 0$ and $m \in \mathbb{N}$.

$\underset{z \neq a}{\overset{\text{for}}{\Longrightarrow}}$ $(z-a)^m f(z) = \frac{(z-a)^m}{h(z)} = \frac{1}{h_1(z)}$

Since this implies $\lim_{z \to a} (z-a)^m f(z)$ exists, this implies, $(z-a)^m f(z)$ has a removable singularity at $z = a$; hence $f(z)$ must be of the form $\frac{g(z)}{(z-a)^m}$ where g is analytic on G.

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