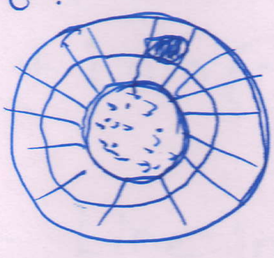
- Circular disk is simply connected but an annulus is not.



- A simply connected domain is one where one can continuously shrink any simple closed curve into a point, while remaining in the domain.

- Indefinite integration of analytic functions

Let $f(z)$ be analytic in a simply connected domain $D$. Then there exists an indefinite integral of $f(z)$ in the domain $D$, that is, an analytic fn. $F(z)$ s.t. $F'(z) = f(z)$ in $D$, & for all paths in $D$ joining two points $z_0$ & $z_1$ in $D$, we have

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0), \quad [F'(z) = f(z)]$$

- Remark: $D$ for an entire function $f$ is the whole complex plane.

Eg ① $\int_0^i z^3\,dz = \left[\frac{z^4}{4}\right]_0^i = \frac{1}{4} - 0 = \frac{1}{4}$.

② $\int_{-\pi i}^{\pi i} \sinh z\,dz = [\cosh z]_{-\pi i}^{\pi i} = \cosh(\pi i) - \cosh(-\pi i) = 0$.

③ $\int_{e^{-\frac{3\pi i}{4}}}^{e^{\frac{3\pi i}{4}}} \frac{1}{z}\,dz = \mathrm{Ln}(e^{\frac{3\pi i}{4}}) - \mathrm{Ln}(e^{-\frac{3\pi i}{4}}) = \frac{3\pi i}{4} - \left(-3\frac{\pi i}{4}\right) = \frac{3\pi i}{2}$

# Use of a representation of a path

- Applies not only to analytic functions but also to any continuous complex function.

**Thm.** Let $C$ be a piecewise smooth path, represented by $z = z(t)$, where $a \le t \le b$. Let $f(z)$ be a continuous function on $C$. Then

$$\boxed{\int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt}, \quad \text{where } \dot{z} = \frac{dz}{dt}$$

**Proof:** Note that, for $z = x+iy$, & $f(z) = u(x(t), y(t)) + iv(x(t), y(t))$

$$\int_C f(z)\,dz = \int_C u\,dx - \int_C v\,dy + i\left(\int_C u\,dy + \int_C v\,dx\right) \quad \text{--- (1)}$$

~~Let~~ Note that $\dot{z} = \dot{x} + i\dot{y}$.

Also $dx = \dot{x}\,dt$ & $dy = \dot{y}\,dt$.

Thus, $\int_a^b f(z(t))\,\dot{z}(t)\,dt = \int_a^b (u+iv)(\dot{x}+i\dot{y})\,dt$

$$= \int_a^b (u+iv)(dx + i\,dy)$$

$$= \int_a^b ((u\,dx - v\,dy) + i(u\,dy + v\,dx))$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx). \quad \text{--- (2)}$$

This proves the result (compare (1) & (2)).

## Steps in applying the above result

1. Represent the path $C$ in the form $z(t)$ $(a \leq t \leq b)$.
2. Calculate the derivative $\dot{z}(t) = \frac{dz}{dt}$.
3. Substitute $z(t)$ for every $z$ in $f(z)$.
4. Integrate $f(z(t))\dot{z}(t)$ over $t$ from $a$ to $b$.

Eg. Prove that $\oint_C \frac{1}{z}\, dz = 2\pi i$, where $C$ is the unit circle traversed in the counter-clockwise direction.

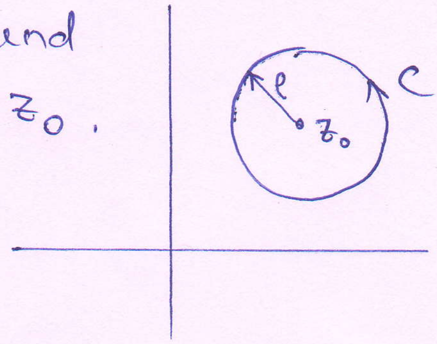
$C$ : unit circle

$$z(t) = \cos t + i \sin t \qquad (0 \leq t \leq 2\pi)$$
$$= e^{it}$$
$$\dot{z}(t) = i e^{it}.$$

$$\Rightarrow \oint_C \frac{1}{z}\, dz = \int_0^{2\pi} e^{-it} \cdot i e^{it}\, dt$$
$$= 2\pi i.$$

# Integral of $\frac{1}{z^m}$ with $m \in \mathbb{Z}$

Let $f(z) = (z - z_0)^m$, $m \in \mathbb{Z}$ & $z_0$, a constant. Integrate $f$ counterclockwise around the circle $C$ of radius $\rho$ and center $z_0$.




**Solution:** Parametrize $C$ as

$$z(t) = z_0 + \rho(\cos t + i\sin t) = z_0 + \rho e^{it} \quad (0 \le t \le 2\pi)$$

Then $(z - z_0)^m = \rho^m e^{imt}$, $dz = i\rho e^{it}\,dt$

$$\Rightarrow \oint_C (z - z_0)^m dz = \int_0^{2\pi} \rho^m e^{imt} i\rho e^{it}\,dt$$

$$= i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t}\,dt$$

$$= i\rho^{m+1} \left( \int_0^{2\pi} \cos(m+1)t\,dt + i \int_0^{2\pi} \sin(m+1)t\,dt \right)$$

When $m = -1$, $\rho^{m+1} = 1$, $\cos 0 = 1$, $\sin 0 = 0$.

& when $m \neq -1$, we have $\int_0^{2\pi} \cos(m+1)t\,dt$

$$= \left[ \frac{\sin(m+1)t}{m+1} \right]_0^{2\pi} = 0$$

Similarly, $\int_0^{2\pi} \sin(m+1)t\,dt = 0$.

Hence

$$\boxed{\oint_C (z - z_0)^m dz = \begin{cases} 2\pi i, & (m = -1), \\ 0, & m \neq -1, m \in \mathbb{Z} \end{cases}}$$

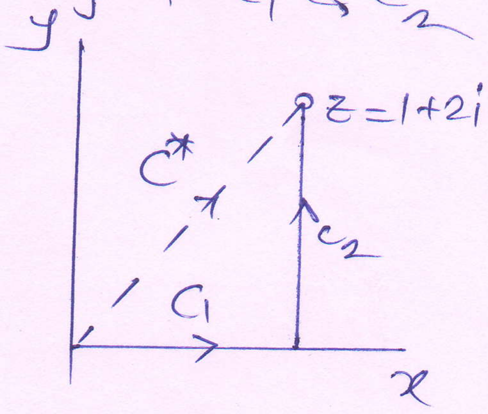
## Path Dependence:

If we integrate a given function $f(z)$ from a point $z_0$ to a point $z_1$ along different paths, the integrals will in general have different values.

- A complex line integral depends not only on the end-points of the path, but, in general, also on the path itself.

**Example:** Integrate $f(z) = Re(z) = x$ from 0 to $1+2i$ along:

(a) $C^*$ (b) $C$ consisting of $C_1$ & $C_2$




(a) $C^* := z(t) = t + 2it \quad (0 \le t \le 1)$.

$\dot{z}(t) = 1 + 2i$,

$f(z(t)) = x(t) = t$

$$\Rightarrow \int_{C^*} Re(z)\,dz = \int_0^1 t(1+2i)\,dt = \frac{1}{2}(1+2i) = \frac{1}{2} + i$$

(b) $C$: $C_1: z(t) = t$, $\dot{z}(t) = 1$, $f(z(t)) = t \quad (0 \le t \le 1)$

$C_2: z(t) = 1 + it$, $\dot{z}(t) = i$, $f(z(t)) = 1 \quad (0 \le t \le 2)$

$$\Rightarrow \int_C Re(z)\,dz = \int_{C_1} Re(z)\,dz + \int_{C_2} Re(z)\,dz = \int_0^1 t\,dt + \int_0^2 i\,dt = \frac{1}{2} + 2i$$