Defn. If f has a pole at $z=a$ and m is the smallest positive integer such that $f(z)(z-a)^m$ has a removable singularity at $z=a$, then f has a pole of order m at $z=a$.

- Suppose m is the order of the pole at $z=a$ and g is analytic s.t. $f(z) = \dfrac{g(z)}{(z-a)^m}$ (where $f: G\setminus\{a\} \to \mathbb{C}$ is analytic and $g: G \to \mathbb{C}$ is analytic), then $g(a) \neq 0$.

- Also, since g is analytic in a disk $B(a;R)$, it has a power series expansion about a, say,

$$g(z) = A_m + A_{m-1}(z-a) + \cdots + A_1(z-a)^{m-1} + (z-a)^m \sum_{k=0}^{\infty} a_k (z-a)^k.$$

Hence,
$$\boxed{f(z) = \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{z-a} + g_1(z),} \qquad (*)$$
where g_1 is analytic in $B(a;R)$ and $A_m \neq 0$.

Defn. If f is a pole of order m at $z=a$ and it satisfies $(*)$, then $\dfrac{A_m}{(z-a)^m} + \dfrac{A_{m-1}}{(z-a)^{m-1}} + \cdots + \dfrac{A_1}{(z-a)}$ is called the singular part of f at $z=a$.

PARTIAL FRACTION EXPANSION OF A RATIONAL FN.

Consider a rational function $r(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials with no common factors.

Then poles of $r(z)$ are exactly the zeros of $q(z)$, and the order of each pole of $r(z)$ is the order of the zero of $q(z)$.

Let $q(a)=0$ and let $S(z)$ be the singular part of $r(z)$ at a. Then $r(z)-S(z)=r_1(z)$, a rational fn. whose poles are also poles of $r(z)$.

- The singular part of $r_1(z)$ at a pole is that of $r(z)$ at that pole.

Hence by induction, if $a_1, \ldots, a_n$ are the poles of $r(z)$ and $S_j(z)$ is the singular part of $r(z)$ at $z=a_j$, then

$$\boxed{r(z) = \sum_{j=1}^{n} S_j(z) + P(z),} \quad \text{—— } (**)$$

with $P(z)$ being a rational function without poles, that is, a polynomial.

- If we allow $P(z)$ to be any analytic function in region G rather than just a polynomial, then $(**)$ is true for any function $r(z)$ analytic in G except for finitely many poles.

<u>Defn.</u> Let $\{z_n : n = 0, \pm 1, \pm 2, \ldots\}$ denote a doubly infinite sequence of complex numbers. Then $\sum_{n=-\infty}^{\infty} z_n$ is said to be absolutely convergent if both $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_{-n}$ are absolutely convergent, and then

$$\sum_{n=-\infty}^{\infty} z_n = \sum_{n=1}^{\infty} z_{-n} + \sum_{n=0}^{\infty} z_n .$$

Now let u_n be a function on a set S for $n \in \mathbb{Z}$, and $\sum_{n=-\infty}^{\infty} u_n(s)$ is absolutely convergent for each $s \in S$, then the convergence is uniform over S if both $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=1}^{\infty} u_{-n}$ converge uniformly on S.

If $\sum_{n=-\infty}^{\infty} z_n$ is abs. conv. with sum z, then $z = \lim_{m\to\infty} \sum_{n=-m}^{m} z_n$.

For $0 \le R_1 < R_2 \le \infty$ and $a \in \mathbb{C}$, define

$$ann(a; R_1, R_2) = \{z : R_1 < |z-a| < R_2\}.$$

- $ann(a; 0, R_2)$ = punctured disk.

LAURENT SERIES EXPANSION OF A FUNCTION ANALYTIC IN AN ANNULUS

If f is analytic in the annulus $ann(a; R_1, R_2)$, then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ where the convergence is absolute and uniform over $\overline{ann(a; r_1, r_2)}$ if $R_1 < r_1 < r_2 < R_2$. Also,

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz, \quad \text{---} (1)$$

where γ is the circle $|z-a| = r$ for any $r \ni R_1 < r < R_2$. Moreover, this Laurent series development of f is unique.

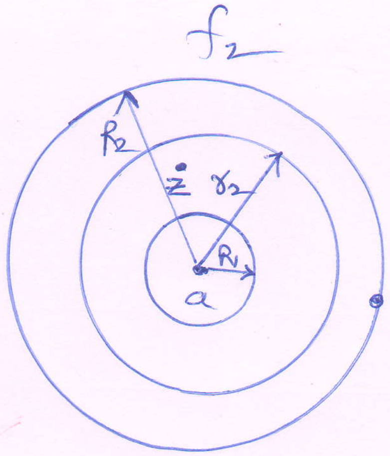
Proof: Let $R_1 < r_1 < r_2 < R_2$; $\gamma_1 : |z-a| = r_1$, $\gamma_2 : |z-a| = r_2$. Then clearly, $\gamma_1 \sim \gamma_2$ in $ann(a; R_1, R_2)$. Thus for any g analytic in $ann(a; R_1, R_2)$, $\int_{\gamma_1} g = \int_{\gamma_2} g$. So the integral in (1) is independent of γ; thus a_n depends only on n, and thus for each integer n, a_n is a constant.

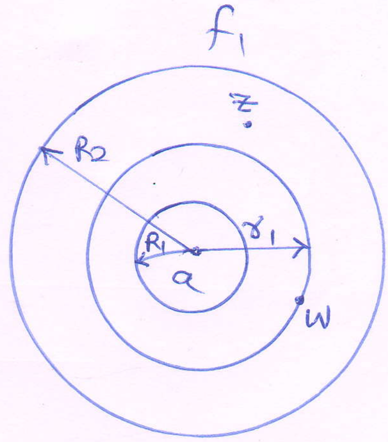
Now the function $f_2 : B(a; R_2) \to \mathbb{C}$ given by

$$f_2(z) = \frac{1}{2\pi i} \int_{|w-a| = r_2} \frac{f(w)}{w-z}\, dw, \text{ where } |z-a| < r_2,\ R_1 < r_2 < R_2,$$

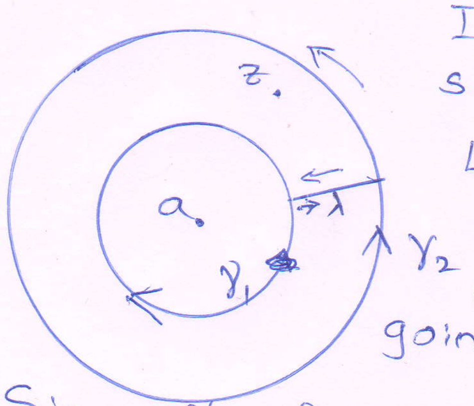
is a well-defined function (independent of r_2).

By one of lemmas (Lemma 5.3) proved in lecture 3, f_2 is analytic in $B(a; R_2)$





Similarly, if $G = \{z : |z-a| > R_1\}$, then $f_1 : G \to \mathbb{C}$ defined by $f_1(z) = \dfrac{-1}{2\pi i} \displaystyle\int_{|w-a|=r_1} \frac{f(w)}{w-z}\,dw$, where $|z-a| > r_1$ & $R_1 < r_1 < R_2$ is analytic in G.



If $R_1 < |z-a| < R_2$, choose r_1 and r_2 s.t. $R_1 < r_1 < |z-a| < r_2 < R_2$.

Let $\gamma_1(t) = a + r_1 e^{it}$, $\gamma_2(t) = a + r_2 e^{it}$ where $0 \le t \le 2\pi$.

Choose a straight line seg. λ radially going from a point on γ_1 to γ_2 missing z.

Since $\gamma_1 \sim \gamma_2$ in $\text{ann}(a; R_1, R_2)$, the closed curve $\gamma := \gamma_2 - \lambda - \gamma_1 + \lambda$ is homotopic to zero in $\text{ann}(a; R_1, R_2)$.

Also $n(\gamma_2, z) = 1$ and $n(\gamma_1; z) = 0$. Hence

$n(\gamma; z) = 1 - 0 = 1$ so that

$$f(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}\,dw$$

$$= f_2(z) + f_1(z).$$

Now f_2 is analytic in $B(a; R_2)$, so it has a power series expansion about a, namely,

$$f_2(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \text{ where } a_n = \int_{\gamma_2} \frac{f(z)}{(z-a)^{n+1}}\,dz. \qquad (2)$$