Bounds for integrals : ML - inequality

$$\left| \int_C f(z)\,dz \right| \le ML$$

where L : Length of C

M : a constant s.t. $|f(z)| \le M$ everywhere on C.

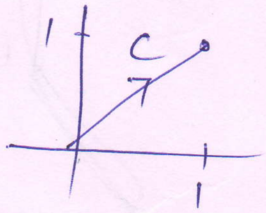
Proof: $$|S_n| = \left| \sum_{m=1}^{n} f(\zeta_m)\,\Delta z_m \right|$$
$$\le \sum_{m=1}^{n} |f(\zeta_m)|\,|\Delta z_m|$$
$$\le M \sum_{m=1}^{n} |\Delta z_m|$$

$|\Delta z_m|$: length of the chord whose endpoints are z_{m-1} and z_m.

$\Rightarrow \sum_{m=1}^{n} |\Delta z_m|$ is the length L^* of the broken line of chords whose end points are $z_0, \cdots, z_n (= Z)$.

Now as $n \to \infty$, $|\Delta t_m|$ & thus $|\Delta z_m| \to 0$ & so $L^* \to L$.

Eg:



$\int_C z^2\,dz$ C : st. line sgmt from 0 to 1+i.

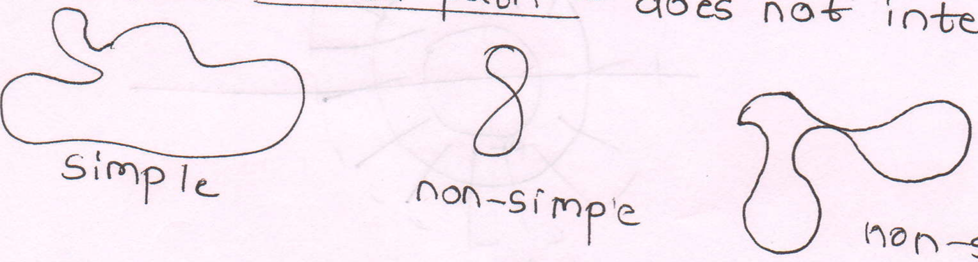
$L = \sqrt{2}$ and $|f(z)| = |z^2| \le 2$

$$\Rightarrow \left| \int_C z^2\,dz \right| \le 2\sqrt{2}.$$

Cauchy's integral theorem

- In general, a line integral of a function $f(z)$ depends not only on the endpoints of the path, but also on the choice of the path itself.
- However, if $f(z)$ is analytic in a simply connected domain D, then the integral is path-independent.

- Simple closed path – does not intersect or touch itself.



simple non-simple non-simple

- Simply connected domain – defn. already seen

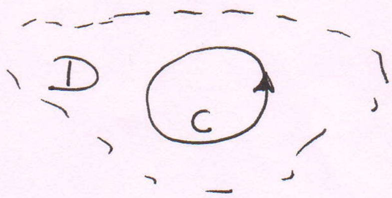
doubly connected

triply connected

Thm. (Cauchy's integral theorem)

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\oint_C f(z)\,dz = 0 .$$



[simple closed path = contour]

Examples ① $\oint_C e^z dz = 0$, $\oint_C \sin z \, dz = 0$, $\oint_C z^n dz = 0$
$n \in \mathbb{N} \cup \{0\}$.

② Singularities outside contour

- $\oint_C \sec z \, dz = 0$ C: unit circle

- $\oint_C \frac{dz}{z^2+9} = 0$ C: unit circle.

③ Non-analytic fn.

$$\oint_C \bar{z} \, dz = \int e^{-it} \cdot e^{it} \cdot i \, dt = 2\pi i \neq 0.$$
(unit circle)

④ Analyticity sufficient, not necessary

$\oint_C \frac{dz}{z^2} = 0$, where C is unit circle.

⑤ Simple connectedness essential

$$\oint_C \frac{1}{z} dz = 2\pi i.$$
C: unit circle. (counter-clockwise)

CAUCHY'S PROOF OF THE ABOVE THEOREM
(with additional assumption on continuity of $f'(z)$)

$$\oint_C f(z) dz = \oint_C (u\,dx - v\,dy) + i \oint_C (u\,dy + v\,dx).$$

$f(z)$ analytic in D $\Rightarrow$ $f'(z)$ exists in D.

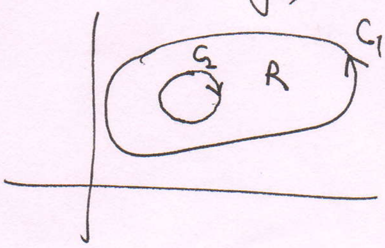
$f'(z)$ continuous $\Rightarrow$ u & v have continuous partial derivatives.

This follows from the fact that

$$f'(z) = u_x + iv_x \quad \& \quad f'(z) = -iu_y + v_y.$$

Then we apply Green's theorem which states that if R is a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves, and if $F_1(x,y)$ and $F_2(x,y)$ are functions that are continuous and have continuous partial derivatives $\frac{\partial F_1}{\partial y}$ & $\frac{\partial F_2}{\partial x}$ every-where in some domain containing R, then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C (F_1\,dx + F_2\,dy).$$



Let $F_1 = u$ & $F_2 = -v$. Then

$$\oint_C (u\,dx - v\,dy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

(region bounded by C)

But $v_x = -u_y \Rightarrow \oint_C (u\,dx - v\,dy) = 0$.

Similarly $u_x = v_y \Rightarrow \oint_C (u\,dy + v\,dx) = 0$.

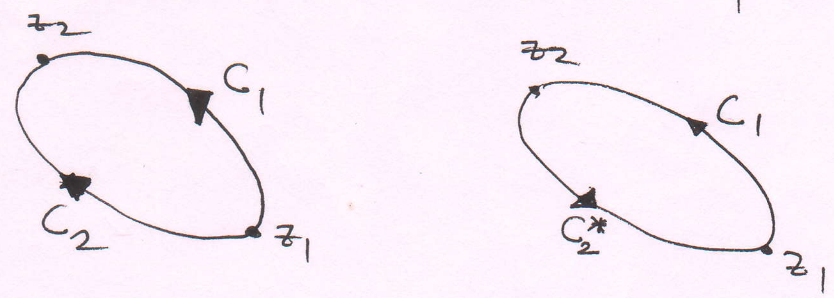
$$\Rightarrow \oint_C f(z)\,dz = 0.$$

Remark: Goursat proved Cauchy's theorem without assuming that $f'(z)$ is continuous.

PATH INDEPENDENCE

<u>Thm.</u> If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

<u>Proof:</u>

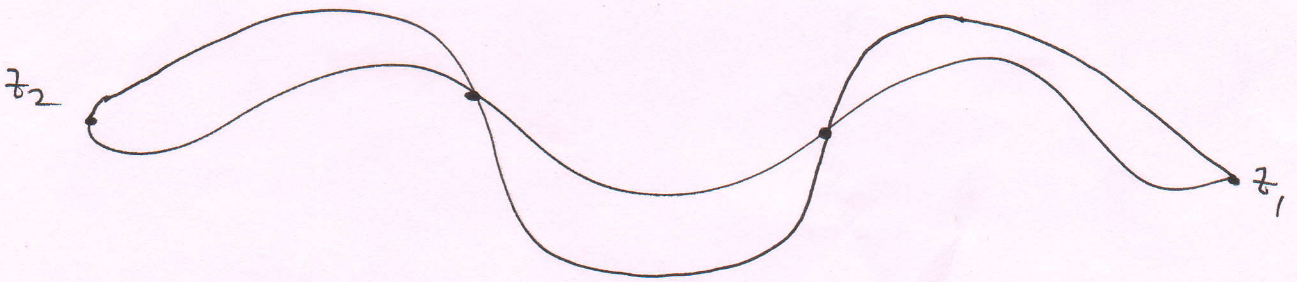


$$\int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz = -\int_{C_2^*} f(z)\,dz.$$

$$= \int_{C_2} f(z)\,dz. \quad \blacksquare$$

<u>Generalization</u>



- <u>Principle of deformation of path</u>

Let D be simply connected & f analytic on D. The line integral of $f(z)$ over a path between points z_0 & z_1 in D retains the same value when we continuously deform it within D keeping z_0 & z_1 fixed.

- Cauchy's theorem for multiply connected domains

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Principle of deformation of Path:

This idea is related to path independence. We may

imagine that the path C_2 is obtained from C_1 ~~be~~ by continuously moving C_1 (with ends fixed) until it coincides with C_2. The integrals along these paths remain unchanged. Hence, we may conduct a continuous deformation of path of an integral, keeping ends fixed. So long as the deforming path always contains points at which $f(z)$ is analytic (inside a simply connected domain) the integral retains the same value; this is called the principle of deformation of path.

Example : Using the principle we can show:

$$\oint_C (z-z_0)^m dz = \begin{cases} 2\pi i & (m=-1) \\ 0 & (m \neq -1) \end{cases}$$

m integer

for counterclockwise integration around any simple closed path C, containing z_0 in its interior.

In fact, for some ~~small~~ $r>0$, the circle, $|z-z_0|=r$ can be continuously deformed in two steps, into a path, just indicated, by first deforming one semicircle and then the other.