Now let $g(z) := f_1\left(a+\frac{1}{z}\right)$ for $0<|z|<\frac{1}{R_1}$.

Note that $z=0$ is an isolated singularity of g.

(Observe that $f_1\left(a+\frac{1}{z}\right)$ is analytic in $\left\{z : |z|<\frac{1}{R_1}\right\}\setminus\{0\}$)

Also $a+1/z \in \operatorname{ann}(a; R_1, \infty)$.

<u>Claim</u>: $z=0$ is a removable singularity of g.

Suppose $R_1<r<R_2$, $z\in \operatorname{ann}(a; r, \infty)$ and $\rho(z)=d(z, C)$, where $C : |w-a|=r$ (circle).

Put $M = \max\{|f(w)| : w\in C\}$.

Then $|f_1(z)| = \left|\frac{-1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{w-z}\,dw\right|$

$= \left|\frac{-1}{2\pi i}\int_{C}\frac{f(w)}{w-z}\,dw\right|$ ($\because \gamma_1 \sim C$ on $\operatorname{ann}(a; R_1, R_2)$)

$\leq \frac{1}{2\pi}\int_C \frac{|f(w)|}{|w-z|}\,|dw| \leq \frac{1}{2\pi}\cdot\frac{M(2\pi r)}{\rho(z)} = \frac{Mr}{\rho(z)}$.

But $\lim_{z\to\infty}\rho(z)=\infty$ so $\lim_{z\to\infty} f_1(z) = 0$.

$\Rightarrow \lim_{z\to 0} g(z) = \lim_{z\to 0} f_1\left(a+\frac{1}{z}\right) = \lim_{z\to\infty} f_1(z) = 0$.

Hence g has a removable singularity at $z=0$.

If we define $g(0)=0$, then g is analytic in $B\left(0; \frac{1}{R_1}\right)$.

$\Rightarrow g(z) = \sum_{n=0}^{\infty} B_n z^n = \sum_{n=1}^{\infty} B_n z^n$ ($\because g(0)=0$)

<u>Claim</u>: $f_1(z) = \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}$, where

a_{-n} is given by $a_n = \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{(w-a)^{n+1}}\,dw$ for $n\leq -1$

(Exercise – Tutorial problem)

3

Note that by defn, $f_1(z) = \frac{-1}{2\pi i}\int_{\gamma_1} \frac{f(w)\,dw}{w-z}$ for $|z-a| > R_1$

$(\gamma_1 : |w-a| = r_1)$

Replacing z by $a+\frac{1}{z}$, we see that for $|z| < \frac{1}{R_1}$,

$$f_1\left(a+\frac{1}{z}\right) = \frac{-1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-\left(a+\frac{1}{z}\right)}\,dw$$

Now employ change of variable $\tilde{w} \Rightarrow \frac{1}{w-a}$ so that $w = a+\frac{1}{\tilde{w}}$. Then $dw = \frac{-1}{\tilde{w}^2}\,d\tilde{w}$.

Also then, γ_1 is to be replaced by $\tilde{\gamma}_1$ given by $\left|a+\frac{1}{\tilde{w}}-a\right| = r_1$, i.e., $\tilde{w} = \frac{1}{r_1}e^{-it}$, $0 \le t \le 2\pi$.

$$\Rightarrow f_1\left(a+\frac{1}{z}\right) = \frac{-1}{2\pi i}\int_{\tilde{\gamma}_1} \frac{f\left(a+1/\tilde{w}\right)}{\left(\frac{1}{\tilde{w}}-\frac{1}{z}\right)} \frac{-1}{\tilde{w}^2}\,d\tilde{w}$$

$$= \frac{z}{2\pi i}\int_{\tilde{\gamma}_1} \frac{f_1\left(a+\frac{1}{\tilde{w}}\right)}{\tilde{w}(z-\tilde{w})}\,d\tilde{w} = \boxed{\frac{z}{2\pi i}\int_{-\tilde{\gamma}_1} \frac{\left(f\left(a+\frac{1}{\tilde{w}}\right)/\tilde{w}\right)}{(\tilde{w}-z)}\,d\tilde{w} = g(z) \text{ on } 0<|z|<\frac{1}{R_1}}$$

where $-\tilde{\gamma}_1(t) = \frac{1}{r_1}e^{it}$ for $0 \le t \le 2\pi$. Again from Lemma 5.3 of lecture 3,

$$g^{(n)}(z) = \frac{n!}{2\pi i}\int_{-\tilde{\gamma}_1} \frac{f\left(a+\frac{1}{\tilde{w}}\right)/\tilde{w}}{(\tilde{w}-z)^n}\,d\tilde{w} + \frac{n!\,z}{2\pi i}\int_{-\tilde{\gamma}_1} \frac{f\left(a+\frac{1}{\tilde{w}}\right)/\tilde{w}}{(\tilde{w}-z)^{n+1}}\,d\tilde{w}$$

on $|z| < R_1$. Thus, $g^{(n)}(0) = \frac{n!}{2\pi i}\int_{-\tilde{\gamma}_1} \frac{f(a+1/\tilde{w})}{\tilde{w}^{n+1}}\,d\tilde{w}$ & so

$$\boxed{B_n = \frac{g^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_{-\tilde{\gamma}_1} \frac{f(a+1/\tilde{w})}{\tilde{w}^{n+1}}\,d\tilde{w}.}$$

Again replace $\tilde{w}$ by $\frac{1}{w-a}$ so that

$$B_n = \frac{1}{2\pi i}\int_{-\gamma_1} \frac{f(w)}{\frac{1}{(w-a)^{n+1}}} \frac{-1}{(w-a)^2}\,dw = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{(w-a)^{-n+1}}\,dw$$

Defining $a_{-n} = B_n$ ~~and replacing n by~~ $-n$, we have

$$a_{-n} = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{(w-a)^{-n+1}}\,dw \text{ for } n \ge 1 \Rightarrow f_1\left(a+\frac{1}{z}\right) = \sum_{n=1}^{\infty} a_{-n} z^n$$

$$\Rightarrow \boxed{f_1(z) = \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} = \sum_{n=-\infty}^{-1} a_n (z-a)^n,}$$

$$\Rightarrow f(z) = f_1(z) + f_2(z)$$

$$= \underbrace{\sum_{n=-\infty}^{-1} a_n (z-a)^n}_{|z-a|>R_1} + \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{|z-a|<R_2}$$

$$= \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad \text{for } z \in \operatorname{ann}(a; R_1, R_2).$$

with $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz$. $\gamma : |z-a| = r$ where $R_1 < r < R_2$.

By (2) & (3), $\sum_{n=-\infty}^{\infty} a_n (z-a)^n$ converges absolutely & uniformly on properly smaller annuli ($\frac{\text{unif. conv. on}}{\operatorname{ann}(a; r_1, r_2)}$ $R_1 < r_1, < r_2 < R_2$)

The uniqueness is derived from the fact that if $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ converges abs. & unif. on proper annuli, then a_n is given by $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz$.

<u>Remark</u>: (1) Note that $\sum_{n=0}^{\infty} a_n (z-a)^n$ converges absolutely for $|z-a| < R_2$ and uniformly for $|z-a| \le r_2$ for $0 < r_2 < R_2$

Similarly, $g(z) = \sum_{n=1}^{\infty} B_n z^n$ converges absolutely for $|z-a| < \frac{1}{R_1}$ and uniformly for $|z-a| \le \frac{1}{r_1}$ for any $r_1 > R_1$.

Thus $f_1(z) = \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$ converges absolutely for $|z-a| > R_1$ and uniformly for $|z-a| \ge r_1$.

---

(2) The details for proving that Laurent series expansion of an analytic function in an annulus is unique are now given.

Let γ_1 and γ_2 be such that $R_1 < \gamma_1 < \gamma_2 < R_2$, & let $\gamma(t) = \left(\frac{\gamma_1+\gamma_2}{2}\right) e^{it}$, $0 \le t \le 2\pi$. We know that the Laurent series converges uniformly on $\overline{\text{ann}(a;\gamma_1,\gamma_2)}$ & $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$.

Case 1: $k \ge 0$.

$$\int_\gamma \frac{f(w)}{(w-a)^{k+1}} dw = \int_\gamma \sum_{n=-\infty}^{\infty} a_n (w-a)^{n-(k+1)} dw$$

$$= \int_\gamma \left[\sum_{n=-\infty}^{-1} a_n (w-a)^{n-(k+1)} + \sum_{n=0}^{\infty} a_n (w-a)^{n-(k+1)} \right] dw \quad \left(\begin{array}{c}\text{by} \\ \text{Defn. of} \\ \text{abs. conv.}\end{array}\right)$$

$$= \int_\gamma \sum_{n=-\infty}^{-1} a_n (w-a)^{n-(k+1)} dw + \int_\gamma \sum_{n=0}^{\infty} a_n (w-a)^{n-(k+1)} dw$$

$$= \sum_{n=1}^{\infty} a_{-n} \int_\gamma \frac{1}{(w-a)^{n+k+1}} dw + \sum_{n=0}^{\infty} a_n \int_\gamma \frac{dw}{(w-a)^{-n+k+1}}$$

(by uniform conv.)

$$= \underset{\substack{\uparrow \\ \text{first sum}}}{0} + \underset{\substack{\uparrow \\ \text{second sum} \\ (n \neq k)}}{0} + a_k \int_\gamma \frac{dw}{(w-a)}$$

(since each integrand has a primitive on $\text{ann}(a;\gamma_1,\gamma_2)$ for $k \neq n$)

$$= a_k \cdot 2\pi i\, n(\gamma; a) = 2\pi i\, a_k \implies \boxed{a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(w)\,dw}{(w-a)^{k+1}}}$$

~~Case~~ Note that $\gamma \sim \tilde{\gamma}$ for any $\tilde{\gamma}$ as long as both are in $\overline{\text{ann}(a;\gamma_1,\gamma_2)}$.

Case 2 corresponding to $k \le -1$ can be similarly proved.