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Principle of deformation of Path:

This idea is related to path independence. We may imagine that the path C_2 is obtained from C_1 ~~be~~ by continuously moving C_1 (with ends fixed)

until it coincides with C_2. The integrals along these paths remain unchanged. Hence, we may conduct a continuous deformation of path of an integral, keeping ends fixed. So long as the deforming path always contains points at which $f(z)$ is analytic (inside a simply connected domain) the integral retains the same value; this is called the principle of deformation of path.

<u>Example</u>: Using the principle we can show:

$$\oint_C (z-z_0)^m \, dz = \begin{cases} 2\pi i & (m=-1) \\ 0 & (m \neq -1) \\ & m \text{ integer} \end{cases}$$

for counterclockwise integration around any simple closed path C, containing z_0 in its interior.

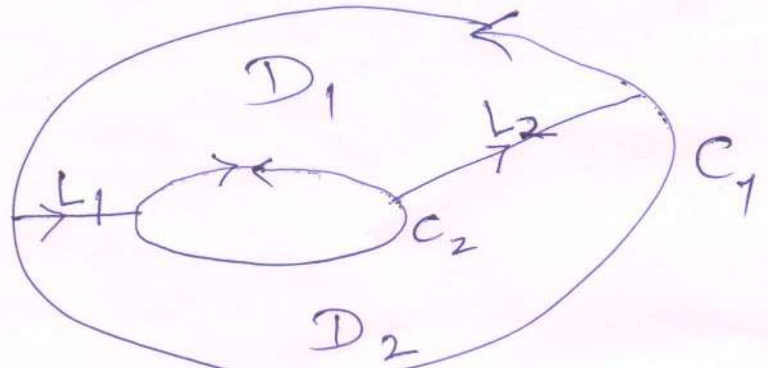
In fact, for ~~small~~ some $r>0$, the circle, $|z-z_0|=r$ can be continuously deformed in two steps, into a path, just indicated, by first deforming one semicircle and then the other.

Cauchy's integral theorem for multiply connected domains:

Cauchy's theorem applies to multiply connected domains. We first explain this for a doubly connected domain D with outer boundary C_1 and inner boundary C_2.

If a function $f(z)$ is analytic in any domain D^* that contains D and its boundary curves, we claim that, $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$, both integrals taken counterclockwise; here full interior of C_2 may not belong to D^*.

<u>Proof</u>:



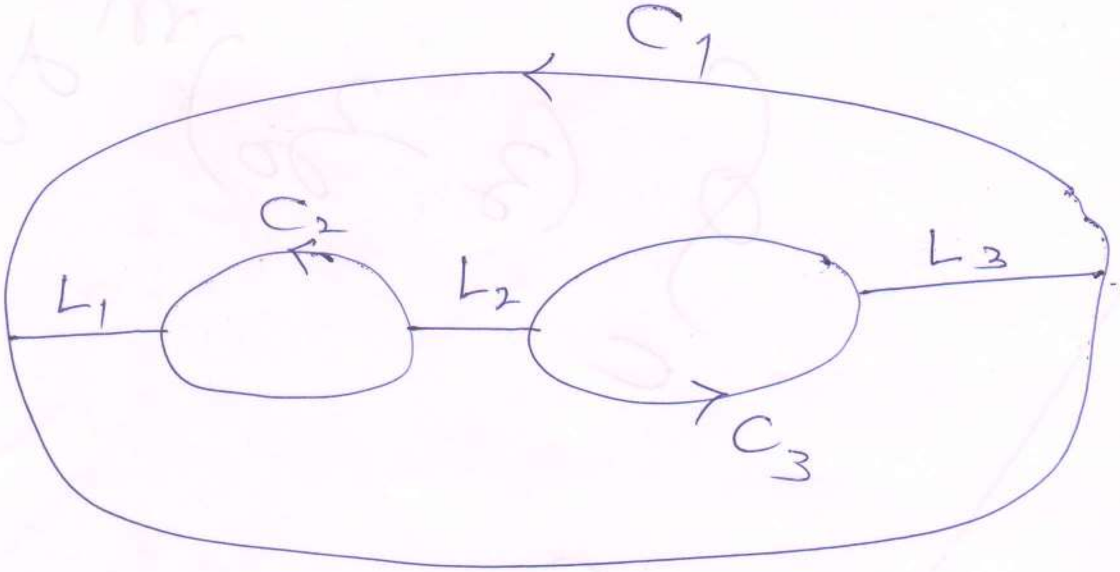
By two cuts L_1 and L_2 we split D into two simply connected domains D_1 and D_2 in which and on whose boundaries, $f(z)$ is analytic. By Cauchy's integral theorem, the integral over the entire boundary of D_1 (taken in the sense of arrows of figure) is zero and so is the integral over the boundary of D_2.

Hence the sum of these integrals is zero. In this sum the integrals along the cuts L_1 and L_2 cancel, since we integrate along them in opposite directions; we are left with the integral along C_1 (counter clockwise) and along C_2 (clockwise). Hence, reversing the integration along C_2, we find:

$$\oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz = 0.$$

This completes the proof.

For domains of higher connectivity idea remains the same. Thus, for a triply connected domain, we use three cuts



Adding integrals as before, the integrals over cuts cancel and the integrals over C_1 (counter-clockwise) and over C_2, C_3 (clockwise) have sum equal to zero. Hence the integral over C_1 equals the sum of integrals over C_2 and C_3, all taken counter clock-wise.

Revisit : $\oint_C (z-z_0)^m dz$.

Cauchy's Integral formula:

The formula is used for evaluating integrals of certain type. It has good consequences also: analytic functions have derivatives of all orders.

Therem: Let $f(z)$ be be analytic in a simply connected domain D. If z_0 is any point in D and C is any simple closed path in D which encloses z_0, then

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i \cdot f(z_0)$$

where integration is taken counter-clockwise.

We may rewrite the formula:

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz.$$

This is Cauchy's integral formula.

Proof depends on Cauchy's integral theorem, some inequalities and ML-inequality.

<u>Examples:</u>

(1) $\oint_C \left(\frac{\sin z}{z - \frac{\pi}{2}} \right) dz$

$= 2\pi i \cdot \sin \frac{\pi}{2} = 2\pi i$,

for any counterclockwise contour C, enclosing $\frac{\pi}{2}$, since $\sin z$ is entire; if C has $\frac{\pi}{2}$ outside it, value is zero.

(2) $\displaystyle\oint_C \frac{3z-4}{2z^2-7z+3}\,dz$, $C : |z|=2$ counterclockwise

$$= \oint_C \left(\frac{1}{2z-1} + \frac{1}{z-3} \right) dz$$

$$= \oint_C \left(\frac{1}{2z-1} \right) dz + \oint_C \left(\frac{1}{z-3} \right) dz$$

$$= \frac{1}{2} \oint_C \left(\frac{1}{z-\frac{1}{2}} \right) dz + \oint_C \left(\frac{1}{z-3} \right) dz$$

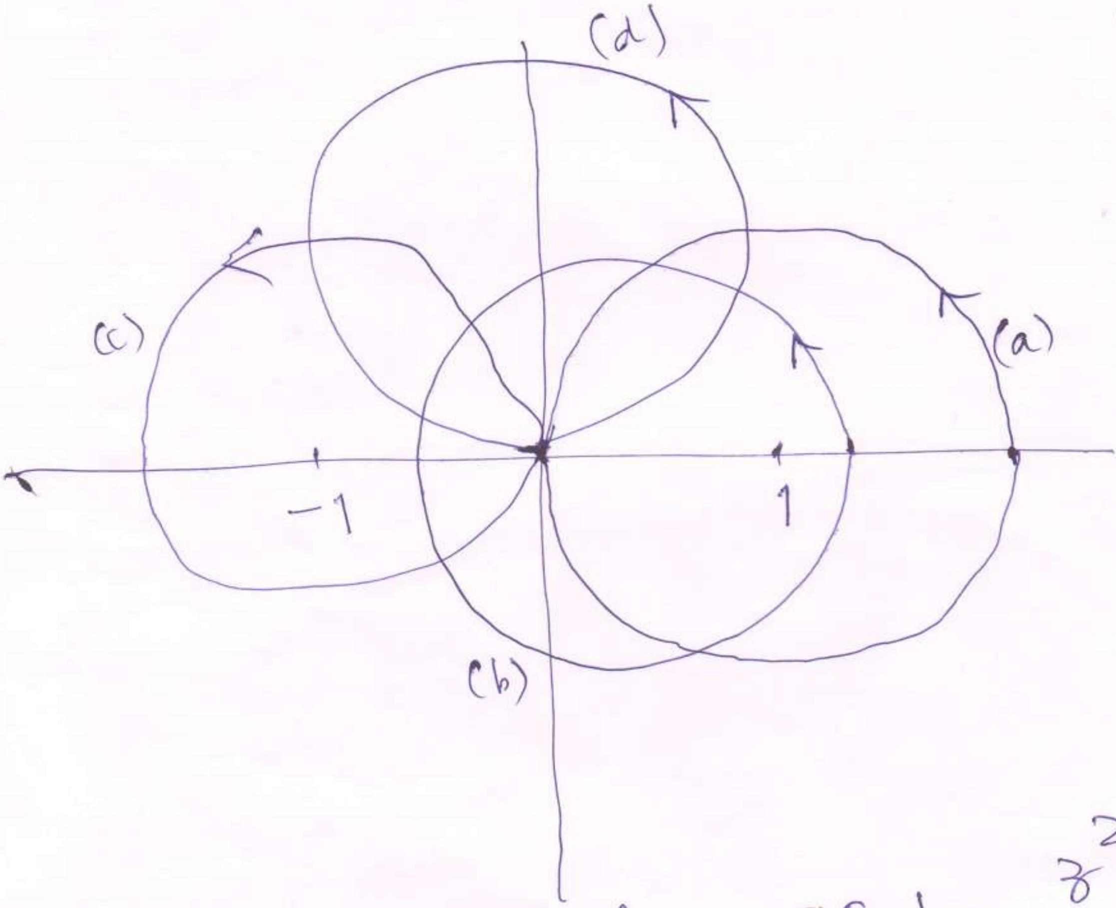
$$= \frac{1}{2} \cdot 2\pi i \cdot 1 + 0 = \pi i .$$

(First integral has $\frac{1}{2}$ enclosed by C, hence Cauchy's integral formula, applies with $f(z)=1, \forall z$.

Second integral has 3 lying outside C, hence Cauchy's integral theorem applies: $\frac{1}{z-3}$ is analytic inside the simply connected domain $D : |z| < \frac{5}{2}$, in which C lies.)

(3) Integrate $g(z) = \dfrac{z^2+1}{z^2-1}$

counterclock-wise around each of the four 'circles' in



the figure here.

Note that $g(z) = \dfrac{z^2+1}{(z-1)(z+1)}$,

so $g(z)$ is analytic everywhere in $\mathbb{C}$ except at $z=1$ & $z=-1$.

We take care to keep these points in our mind. Let us Consider each circle separately:

(a) The circle, $|z-1|=1$ encloses the point $z_0=1$ (only) where $g(z)$ is not analytic. Hence, we represent $g(z)$ as

$$g(z) = \left(\frac{z^2+1}{z+1}\right) \cdot \frac{1}{z-1} = \frac{f(z)}{z-1}$$

Where $f(z) = \frac{z^2+1}{z+1}$, is analytic in some simply connected domain, containing the circle $|z-1|=1$. Thus

$$\oint_C \left(\frac{z^2+1}{z^2-1}\right) dz = 2\pi i \cdot f(1) = 2\pi i.$$

(b) Answer is same as (a) by principle of deformation of path.

(c) The function $g(z)$ is same as before; since the 'circle' (simple, closed path) encloses $z_0 = -1$ (only), we may express $g(z)$ as $g(z) = \frac{h(z)}{z+1}$

where $h(z) = \frac{z^2+1}{z-1}$.

Thus: $\oint_C g(z)\,dz = \oint_C \left(\frac{z^2+1}{z-1}\right) \cdot \frac{1}{z+1}\,dz$

$= \oint_C \frac{h(z)}{z+1}\,dz = 2\pi i \cdot h(-1)$

$= 2\pi i(-1) = -2\pi i$.

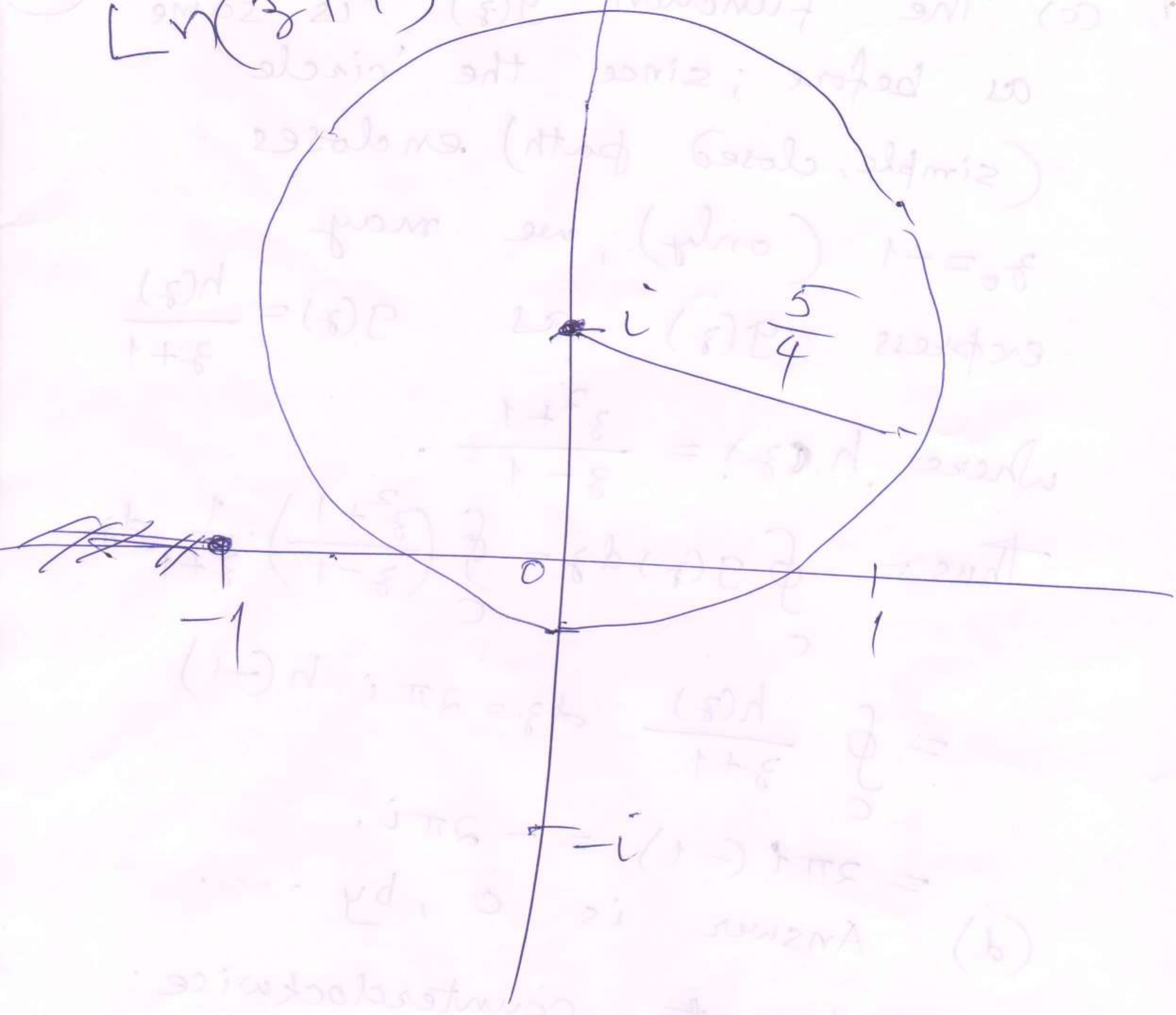
(d) Answer is 0, by ....

(4) Integrate counterclockwise:

$$\int_C \frac{\operatorname{Ln}(z+1)}{z^2+1}, \quad C: |z-i| = \frac{5}{4}.$$

Ans: $\left(\frac{\operatorname{Ln}(i+1)}{i+i}\right) \cdot 2\pi i = \pi\left(\ln\sqrt{2} + \frac{i\pi}{4}\right)$.

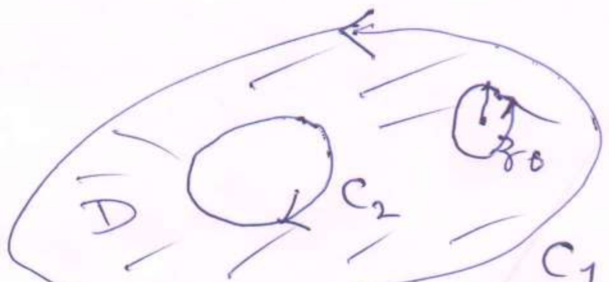
$Ln(z+1)$



$$|z-i| = \frac{5}{4}$$

Multiply connected domains can be treated as in discussion relating to Cauchy's integral theorem for such domains:

If $f(z)$ is analytic on paths C_1, C_2 and in the ring shaped domain bounded by them, as shown in adjoining figure.



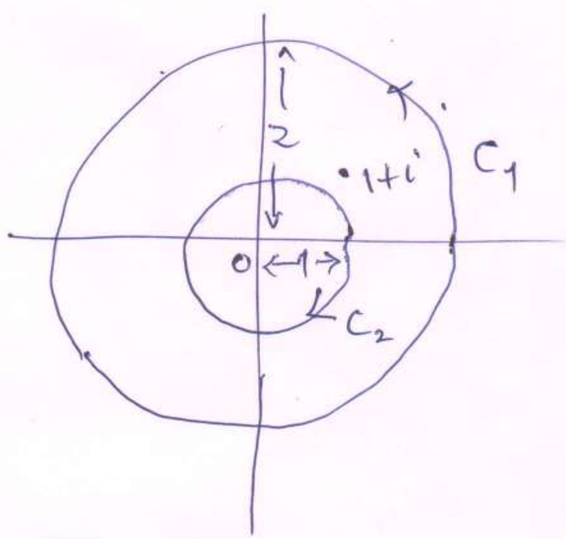
If z_0 is any point in the domain D, then

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z - z_0}\, dz + \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{z - z_0}\, dz,$$

where the outer integral over C_1 is taken counterclockwise and the inner clockwise, as indicated in figure.

Ex: If C_1 is $|z|=2$, counterclockwise and C_2 is $|z|=1$ clockwise, then

$$\oint_{C_1} \frac{e^{z^2}}{z^2(z-1-i)}\,dz + \oint_{C_2} \frac{e^{z^2}}{z^2(z-1-i)}\,dz$$



$$= 2\pi i \left[\left. \frac{e^{z^2}}{z^2} \right|_{z=1+i} \right]$$

$$= 2\pi i \left[\frac{e^{2i}}{2i} \right]$$

$$= \pi e^{2i}.$$

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Proof of Cauchy's integral formula

• **Thm.** Let f be analytic in a simply connected domain D. Then for any point z_0 in D & any simple closed path C in D that encloses z_0,

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0).$$

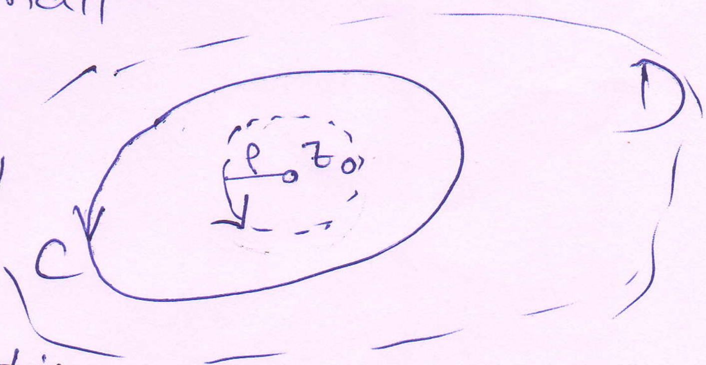
Proof – $f(z) = f(z_0) + (f(z) - f(z_0))$

$$\Rightarrow \oint_C \frac{f(z)\,dz}{z-z_0} = f(z_0)\oint_C \frac{dz}{z-z_0} + \oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$= 2\pi i f(z_0) + \oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz.$$

Goal: To show $\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0$.

Integrand is analytic except possibly at z_0.
Hence we can replace C by a small circle K of radius ρ and center z_0 without altering the value of the integral.



Since f is analytic, it is continuous.
Hence given an $\varepsilon > 0$, $\exists\ \delta > 0 \ni |f(z) - f(z_0)| < \varepsilon$ for all z in the disk $|z - z_0| < \delta$.
Let the radius ρ of $K < \delta$. Then

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \frac{\varepsilon}{\rho}$$

$$\Rightarrow \left|\oint_K \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| < \frac{\varepsilon}{\rho}\cdot 2\pi\rho = 2\pi\varepsilon$$

(ε arbitrary small $\Rightarrow \oint_K \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0$)