MA 502 - SEM. II (PART II) - LECTURE 13

CLASSIFICATION OF ISOLATED SINGULARITIES

Cor. 6.4 Let $z=a$ be an isolated singularity of f and let $f(z)=\sum_{n=-\infty}^{\infty} a_n(z-a)^n$ be its Laurent series expansion in $\mathrm{ann}(a;0,R)$. Then:

a) $z=a$ is a removable singularity iff $a_n=0$ for $n\le -1$,

b) $z=a$ is a pole of order m iff $a_{-m}\neq 0$ and $a_n=0$ for $n\le -(m+1)$,

c) $z=a$ is an essential singularity iff $a_n\neq 0$ for infinitely many negative integers n.

Proof: a) "$\Leftarrow$" Let $a_n=0$ for $n\le -1$. Then $g(z):=\sum_{n=0}^{\infty} a_n(z-a)^n$ defined on $B(a;R)$ is analytic and it agrees with f in the punctured disk.

"$\Rightarrow$" Suppose f has a removable singularity at $z=a$. Then
$$0=\lim_{z\to a}(z-a)f(z)=\lim_{z\to a}\sum_{n=-\infty}^{\infty} a_n(z-a)^{n+1}$$
$$=\sum_{n=-\infty}^{\infty} a_n \lim_{z\to a}(z-a)^{n+1} \quad \text{(uniform convergence in, say, } \mathrm{ann}(a,0,r) \text{ for } 0<r<R)$$

This will be true only if $a_n=0$ for $n\le -1$.

b) "$\Leftarrow$" Suppose $a_n=0$ for $n\le -(m+1)$ and $a_{-m}\neq 0$. Then $(z-a)^m f(z)$ has a Laurent series expansion with no negative powers of $(z-a)$. By part (a), $(z-a)^m f(z)$ has a removable singularity at $z=a$. Thus f has a pole of order m at $z=a$.

"$\Rightarrow$" The steps above are reversible.

(2) c) By defn., c) follows from a) & b). ☒

Remark: f has an essential singularity implies $\lim_{z \to a} |f(z)|$ does not exist. (∞ is included in the existence)
That means $f(z)$ wanders through $\mathbb{C}$ as $z \to a$.

Thm. 6.5 (Casorati - Weierstrass theorem)
If f has an essential singularity at $z=a$, then for every $\delta > 0$ $\overline{f(\text{ann}(a;0;\delta))} = \mathbb{C}$.

Proof:- (by contradiction)
Suppose f is analytic in $\text{ann}(a;0,R)$. We have to show that given $c \in \mathbb{C}$ & $\varepsilon > 0$, for each $\delta > 0$, $\exists$ z with $|z-a| < \delta$ such that $|f(z) - c| < \varepsilon$.

Suppose this is not true, that is, $\exists c \in \mathbb{C}$ & $\varepsilon > 0 \ni$
$|f(z) - c| \geq \varepsilon \ \forall z \in G := \text{ann}(a;0,\delta)$. Then

$$\lim_{z \to a} \left| \frac{f(z) - c}{z - a} \right| = \infty.$$

Hence $\dfrac{f(z)-c}{z-a}$ has a pole at $z=a$. Suppose it is of order m. Then

$$\lim_{z \to a} (z-a)^{m+1}(f(z) - c) = 0 \quad \left(\because (z-a)^m \left(\frac{f(z)-c}{z-a}\right) \text{ has a removable sing. at } z=a \right)$$

$$\Rightarrow \lim_{z \to a} |z-a|^{m+1} |f(z) - c| = 0. \longrightarrow (*)$$

$$\Rightarrow |z-a|^{m+1} |f(z)| = |z-a|^{m+1} |f(z) - c + c|$$
$$\leq |z-a|^{m+1} |f(z) - c| + |z-a|^{m+1} |c|$$

By $(*)$ $\Rightarrow \lim_{z \to a} |z-a|^{m+1} |f(z)| = 0$ since $m \geq 1$.

$\Rightarrow (z-a)^m f(z)$ has a removable singularity at $z=a$

$\Rightarrow$ f has a pole of order m at $z=a$.

$\longrightarrow\!\leftarrow$ since f has an essential singularity at $z=a$.

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3) RESIDUES

Q: If f has an isolated singularity at $z=a$, what are the possible values of $\int_\gamma f$ for γ a closed curve, $\gamma \approx 0$ and not passing through a?

Defn. Let f have an isolated singularity at $z=a$. & let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ be its Laurent series expansion about $z=a$. Then the residue of f at $z=a$ is the coefficient a_{-1}, denoted by $\text{Res}(f;a) = a_{-1}$.

RESIDUE THEOREM (Thm. 6.6)

Let f be analytic in the region G except for the isolated singularities $a_1, a_2, \ldots, a_n$. If γ is a closed rectifiable curve in G which does not pass through any of the points a_k and if $\gamma \approx 0$ in G, then

$$\frac{1}{2\pi i}\int_\gamma f = \sum_{k=1}^{n} n(\gamma; a_k)\, \text{Res}(f; a_k).$$

Proof: Let $m_k := n(\gamma; a_k)$ for $1 \le k \le m$.
Choose positive numbers $r_1, r_2, \ldots r_m$ s.t. the disks $\bar{B}(a_k; r_k)$ are disjoint, none of them intersect $\{\gamma\}$ and each disk is contained in G.
(Note that $\{\gamma\}$ is compact. Also if A and B are disjoint sets in X with B closed and A compact, then $d(A,B) > 0$.) Also $\{\gamma\}$ does not pass through any of the singularities of f.)

Let $\gamma_k(t) = a_k + r_k \exp(-2\pi i m_k t)$ for $0 \le t \le 1$.
Then for $1 \le j \le m$, $n(\gamma; a_j) + \sum_{k=1}^{m} n(\gamma_k; a_j) = 0$
(Note that the orientation of every γ_k is opposite to that of γ by defn.)

4) Note that $n(\gamma_k; a_j) = 0$ for $k \neq j$ and $n(\gamma_k; a_k) = -m_k$
$= -n(\gamma; a_k)$. ———— (1)

Also since $\gamma \approx 0$ in G, by defn., $n(\gamma; a) = 0 \ \forall a \in \mathbb{C} \setminus G$. ———— (2)

Moreover, since ~~B(ak;~~ $\bar{B}(a_k; r_k) \subseteq G$, we have
$n(\gamma_k; a) = 0 \ \forall a \in \mathbb{C} \setminus G$ and for $1 \leq k \leq m$. ———— (3)

From (1), (2) and (3),

$$n(\gamma; a) + \sum_{k=1}^{m} n(\gamma_k; a) = 0$$ for all a not in
$G \setminus \{a_1, a_2, \dots, a_m\} = \mathbb{C} \setminus (G \setminus \{a_1, \dots, a_m\})$.

Since f is analytic in $G \setminus \{a_1, \dots, a_m\}$,

$$0 = \int_{\gamma} f + \sum_{k=1}^{m} \int_{\gamma_k} f. \quad \text{(CAUCHY'S THM. 1st version)} \quad \text{———— (4)}$$

Now if $f(z) = \sum_{n=-\infty}^{\infty} b_n (z - a_k)^n$ is the Laurent series expn. about $z = a_k$, then this series converges uniformly in $\bar{B}(a_k; r_k)$. Hence

$$\int_{\gamma_k} f(z) dz = \int_{\gamma_k} \sum_{n=-\infty}^{\infty} b_n (z - a_k)^n = \sum_{n=-\infty}^{\infty} b_n \int_{\gamma_k} (z - a_k)^n dz$$

For $n \neq -1$, $(z - a_k)^n$ has a primitive $\Rightarrow \int_{\gamma_k} (z - a_k)^n dz = 0$.

For $n = -1$, we have the summand

$$= b_{-1} \int_{\gamma_k} \frac{dz}{(z - a_k)} = \text{Res}(f; a_k) \, 2\pi i \, n(\gamma; a_k)$$

Now from (4),

$$\int_{\gamma} f(z) dz = -\sum_{k=1}^{m} \int_{\gamma_k} f(z) dz$$

$$= -\sum_{k=1}^{m} 2\pi i \, n(\gamma_k; a_k) \, \text{Res}(f; a_k)$$

$$= 2\pi i \sum_{k=1}^{m} n(\gamma; a_k) \, \text{Res}(f; a_k) \quad \left(\because n(\gamma_k; a_k) = -n(\gamma; a_k) \right)$$

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