7) • Suppose f has a pole of order $m \geq 1$ at $z=a$. Then $g(z) = (z-a)^m f(z)$ has a removable singularity at $z=a$ and $g(a) \neq 0$.

Let $g(z) = b_0 + b_1(z-a) + \cdots$ be the power series expansion of g about $z=a$. Then for z near a, $z \neq a$,

$$f(z) = \frac{b_0}{(z-a)^m} + \cdots + \frac{b_{m-1}}{(z-a)} + \sum_{k=0}^{\infty} b_{m+k}(z-a)^k.$$

This is the Laurent series expansion of f in a punctured disk about $z=a$.

But $\text{Res}(f;a) = b_{m-1}$

In particular if $z=a$ is a simple pole, then $m=1$, so, $\text{Res}(f;a) = b_0 = g(a) = \lim_{z \to a} (z-a)f(z)$.

More generally,

Thm 6.6. Suppose f has a pole of order m at $z=a$ and if $g(z) = (z-a)^m f(z)$, then

$$\text{Res}(f;a) = \frac{1}{(m-1)!} g^{(m-1)}(a)$$

Proof: b_{m-1} is the coefficient of $(z-a)^{m-1}$ in the power series representation of g.

$$\Rightarrow b_{m-1} = \frac{g^{(m-1)}(a)}{(m-1)!}$$

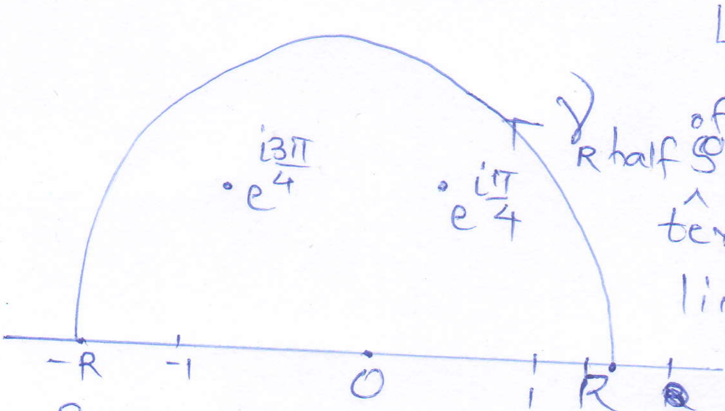
$\triangle$

Evaluating integrals of functions of real variable using the residue theorem

(1) Prove that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}}$

Proof: Let $f(z) = \dfrac{z^2}{1+z^4}$. The poles of f are the 4^{th} roots of (-1), which are $e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{\frac{i5\pi}{4}}, e^{\frac{i7\pi}{4}}$.

These are simple poles.



Let $R>1$ and consider γ to be the closed path formed by upper half of semicircle of radius R centered at zero, and the straight line segment $[-R, R]$, i.e.

$$\gamma = \gamma_R + [-R,R]$$

By the Cauchy residue theorem,

$$\frac{1}{2\pi i}\int_\gamma f = \operatorname{Res}(f; e^{i\pi/4}) + \operatorname{Res}(f; e^{i3\pi/4})$$

$$= \lim_{z\to e^{i\pi/4}} (z-e^{i\pi/4}) f(z) + \lim_{z\to e^{i3\pi/4}} (z-e^{\frac{i3\pi}{4}}) f(z)$$

Note that $\displaystyle\lim_{z\to e^{i\pi/4}} (z-e^{i\pi/4}) f(z)$

$$= \lim_{z\to e^{i\pi/4}} \frac{\cancel{(z-e^{i\pi/4})}\, z^2}{\cancel{(z-e^{i\pi/4})}(z-e^{\frac{i3\pi}{4}})(z-e^{i5\pi/4})(z-e^{i7\pi/4})}$$

$$= \lim_{z\to e^{i\pi/4}} \frac{z^2}{(z-e^{i3\pi/4})(z-e^{i5\pi/4})(z-e^{i7\pi/4})} = \frac{1-i}{4\sqrt{2}}.$$

<u>Another formula</u>: If $f(z) = \dfrac{g(z)}{h(z)}$ is a function with $g, h \in \mathcal{A}(B(a;\delta))$, $g(a) \neq 0$, and h has a simple zero.

at $z=a$, then $\operatorname{Res}(f;a) = \dfrac{g(a)}{h'(a)}$, (Exercise).

Hence using the above result, we see

$$\operatorname{Res}\left(\frac{z^2}{1+z^4}; e^{i\pi/4}\right) = \left.\frac{z^2}{4z^3}\right|_{z=e^{i\pi/4}} = \frac{1}{4e^{i\pi/4}}$$

$$= \frac{1-i}{4\sqrt{2}}.$$

Similarly, $\operatorname{Res}(f; e^{i3\pi/4}) = \dfrac{-1-i}{4\sqrt{2}}$.

$$\Rightarrow \frac{1}{2\pi i}\int_\gamma f = \frac{1-i}{4\sqrt{2}} + \frac{-1-i}{4\sqrt{2}} = \frac{-i}{2\sqrt{2}} \quad \text{—— (1)}$$

Now by the defn. of ~~the~~ line integral,

$$\frac{1}{2\pi i}\int_\gamma f = \frac{1}{2\pi i}\int_{-R}^{R} \frac{x^2}{1+x^4}\,dx + \frac{1}{2\pi i}\int_{\gamma_R} f \quad \text{—— (2)}$$

But
$$\frac{1}{2\pi i}\int_{\gamma_R} f = \frac{1}{2\pi i}\int_0^{\pi} \frac{(Re^{it})^2\, iRe^{it}\,dt}{1+(Re^{it})^4}$$

$$= \frac{R^3}{2\pi}\int_0^{\pi} \frac{e^{i3t}\,dt}{1+R^4e^{i4t}}.$$

For $0 \le t \le \pi$, $1+R^4e^{4it}$ lies on the circle centered at 1 of radius R^4. Hence $|1+R^4e^{4it}| \geq R^4-1$.

(This can also be seen by applying the reverse Δ ineq.

$$|1+R^4e^{4it}| = |R^4e^{4it}-(-1)| \geq \big|\,|R^4e^{4it}| - |-1|\,\big|$$
$$= |R^4-1| = R^4-1$$
(since $R>1$))

4) Hence $\left| \dfrac{R^3}{2\pi} \displaystyle\int_0^{\pi} \dfrac{e^{i3t}\,dt}{1+R^4 e^{i4t}} \right| \leq \dfrac{R^3}{2\pi} \cdot \dfrac{\pi}{R^4-1} \longrightarrow 0$ as $R \to \infty$.

Hence $\displaystyle\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma_R} f = 0$ ——— (3)

From (1), (2) and (3),

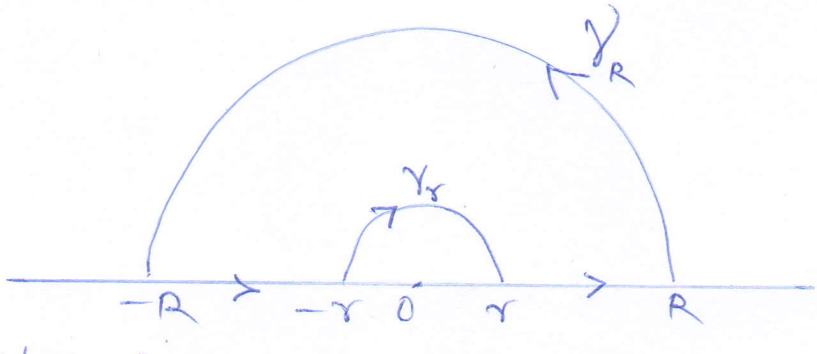
$$\frac{1}{2\pi i}\int_\gamma f = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{-R}^{R} \frac{x^2}{1+x^4}\,dx + 0$$

$$\Rightarrow \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = \frac{-i}{2\sqrt{2}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}}.$$

2) Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

Proof: Let $f(z) = \dfrac{e^{iz}}{z}$. It has a simple pole at $z=0$. Let $0<r<R$ and let γ be the following curve



i.e; $\gamma = \gamma_R + [-R,-r] + \gamma_r + [r,R]$

By Cauchy's thm; $\displaystyle\int_\gamma f = 0$ ($\because f$ is analytic in the region bdd. by γ)

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$$\Rightarrow 0 = \int_{r}^{R} \frac{e^{ix}}{x}\,dx + \int_{\gamma_R} \frac{e^{iz}}{z}\,dz + \int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_{\gamma_r} \frac{e^{iz}}{z}\,dz. \quad \text{—— (1)}$$

$$\left.\begin{aligned}\Rightarrow \int_{r}^{R} \frac{\sin x}{x}\,dx &= \frac{1}{2i}\int_{r}^{R} \frac{e^{ix}-e^{-ix}}{x}\,dx \\ &= \frac{1}{2i}\int_{r}^{R} \frac{e^{ix}}{x}\,dx + \frac{1}{2i}\int_{-R}^{-r} \frac{e^{ix}}{x}\,dx\end{aligned}\right\} \quad (2)$$

Now $\left|\int_{\gamma_R} \frac{e^{iz}}{z}\,dz\right| = \left|i\int_{0}^{\pi} \exp(iRe^{i\theta})\,d\theta\right|$ $\left(\because z = Re^{i\theta} \text{ implies } dz = iRe^{i\theta}d\theta\right)$

$$\le \int_{0}^{\pi} \left|\exp(iRe^{i\theta})\right| d\theta.$$

$$= \int_{0}^{\pi} \exp(-R\sin\theta)\,d\theta = 2\int_{0}^{\pi/2} \exp(-R\sin\theta)\,d\theta$$

$$\le 2\int_{0}^{\pi/2} \exp\left(-\frac{2R\theta}{\pi}\right) d\theta \quad \left(\because \sin\theta \ge 2\theta/\pi \text{ on } [0, \pi/2]\right)$$

$$= \left.\frac{2\exp\left(-\frac{2R\theta}{\pi}\right)}{\left(-2R/\pi\right)}\right|_{\theta=0}^{\theta=\pi/2} = \frac{-\pi}{R}\left(e^{-R}-1\right)$$

$$= \frac{\pi}{R}\left(1-e^{-R}\right).$$

$$\Rightarrow \lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{z}\,dz = 0. \quad \text{—— (3)}$$

Now consider $\int_{\gamma_r} \frac{e^{iz}}{z}\,dz = \int_{\gamma_r} \frac{e^{iz}-1}{z}\,dz + \int_{\gamma_r} \frac{1}{z}\,dz.$

$$= \int_{\gamma_r} \frac{e^{iz}-1}{z}\,dz - \pi i \quad \text{—— (4)}$$

Now $\frac{e^{iz}-1}{z}$ has a removable singularity at $z=0$

$\Rightarrow \exists$ M (constant), $M>0 \ni \left|\frac{e^{iz}-1}{z}\right| \leq M$ for $|z| \leq 1$,

$\Rightarrow \left|\int_{\gamma_r} \frac{e^{iz}-1}{z}\, dz\right| \leq \pi r M$ so that

$$\lim_{r\to 0} \int_{\gamma_r} \frac{e^{iz}-1}{z}\, dz = 0 \quad (5)$$

which gives from (4), $\lim_{r\to 0} \int_{\gamma_r} \frac{e^{iz}}{z}\, dz = -\pi i$

Thus from (1), (2), (3) and (5),

$$\int_0^\infty \frac{\sin x}{x}\, dx = \lim_{R\to\infty} \lim_{r\to 0} \int_r^R \frac{\sin x}{x}\, dx$$

$$= \frac{1}{2i}\left\{-\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{iz}}{z}\, dz - \lim_{r\to 0} \frac{e^{iz}}{z}\, dz\right\}$$

$$= \frac{1}{2i}\left(-0-(-\pi i)\right) = \frac{\pi}{2}.$$

(3) For $a>1$, show that $\int_0^\pi \frac{d\theta}{a+\cos\theta} = \frac{\pi}{\sqrt{a^2-1}}$.

Proof: Let $z=e^{i\theta}$, then $\bar{z}=e^{-i\theta}=1/z$.

and $a+\cos\theta = a+\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right) = a+\frac{1}{2}\left(z+\frac{1}{z}\right)$

$$= \frac{z^2+2az+1}{2z},$$