Section 14.4 – Derivatives of Analytic functions

Thm. 1 If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are analytic functions in D too. The values of these derivatives at a point z_0 in D are given by

$$\boxed{\begin{aligned} f'(z_0) &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz, \\ f''(z_0) &= \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz, \end{aligned}}$$

and, in general,

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (n \in \mathbb{N}),}$$

where C is any simple closed curve (path) in D enclosing z_0 and whose complete interior belongs to D. We integrate in a ~~[illegible]~~ counter-clockwise direction.

Examples ① $\displaystyle\oint_{\substack{C \\ \text{(unit circle)}}} \frac{e^{-z}\sin z}{z^2}\,dz = \oint_C \frac{e^{-z}\sin z}{(z-0)^2}\,dz$

$= 2\pi i f'(0)$, where

$f(z) = e^{-z}\sin z$.

Hence $f'(z) = e^{-z}(\cos z - \sin z)$ so that $f'(0) = 1$

$\Rightarrow \displaystyle\oint_C \frac{e^{-z}\sin z}{z^2}\,dz = 2\pi i$.

(2) $\oint_C \frac{z^6}{(2z-1)^6}\,dz = \frac{1}{2^6}\oint_C \frac{z^6}{(z-\frac{1}{2})^6}\,dz$
(unit circle)

$= \frac{1}{2^6}\cdot\frac{2\pi i}{5!}\, f^{(5)}(z)\Big|_{z=\frac{1}{2}}$, where $f(z) = z^6$.

$f^{(5)}(z) = 6!\,z \implies f^{(5)}(z)\Big|_{z=1/2} = \frac{6!}{2}$ so that

$$\oint_C \frac{z^6}{(2z-1)^6}\,dz = \frac{3\pi i}{32}.$$

(3) $\oint_C \frac{\mathrm{Ln}(z+3)}{(z-2)(z+1)^2}\,dz$ (C: boundary of the square with vertices ± 1.5, $\pm 1.5i$)

$= 2\pi i\, f'(-1)$, where

$f(z) = \frac{\mathrm{Ln}(z+3)}{z-2}$,

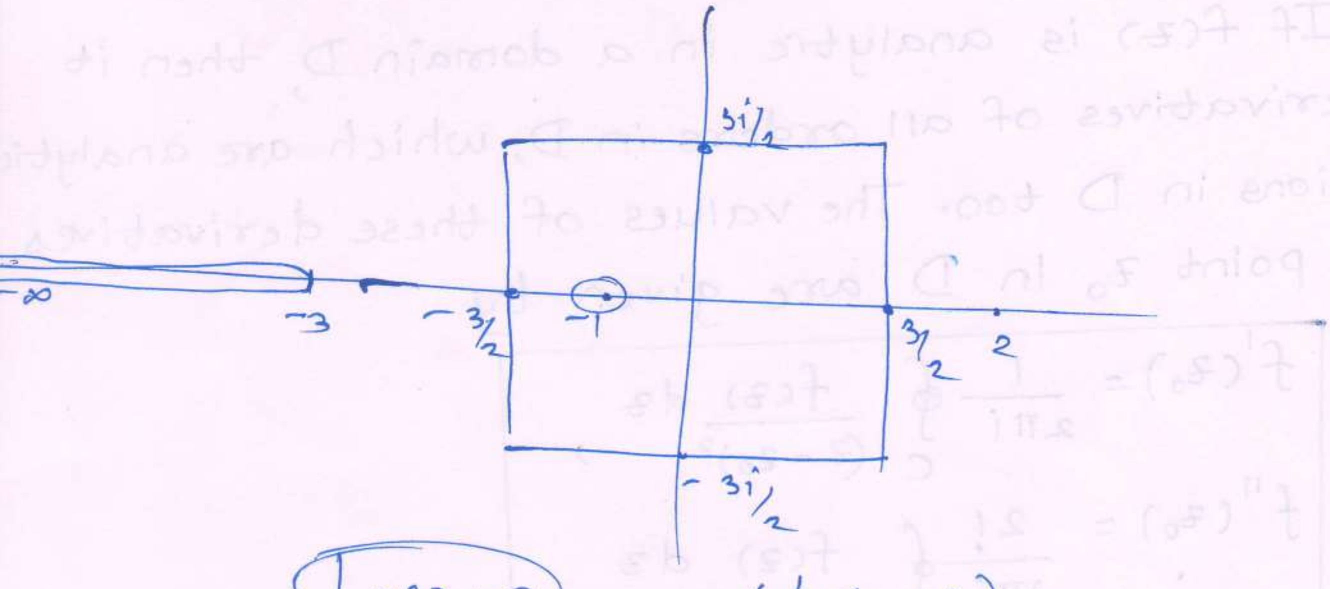
$$\boxed{\text{Ans. } 2\pi i\left(-\frac{1}{6} - \frac{\mathrm{Ln}(2)}{9}\right)}$$

Corollary

Cauchy's inequality

If $f(z)$ is analytic in domain D, and C is a ~~simple closed path~~ circle of radius r enclosing $z_0 \in D$ such that $|f(z)| \le M$ for all z on C, and $M > 0$, then

$$\boxed{|f^{(n)}(z_0)| \le \frac{n!\,M}{r^n}.}$$



$$\frac{Ln(z+3)}{(z-2)(z+1)^2} = \frac{\left(\dfrac{Ln(z+3)}{z-2}\right)}{(z-(-1))^2}$$

Proof: By Thm. 1,

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

$$\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(z)|}{r^{n+1}} . |ire^{i\theta} d\theta| \qquad (z = z_0 + re^{i\theta})$$

$$\leq \frac{n!}{2\pi} \cdot \frac{M}{r^n} \cdot \int_0^{2\pi} 1 \, d\theta$$

$$= \frac{n! M}{r^n}.$$

$\boxed{\text{Thm. 2}}$ $\boxed{\text{Liouville's theorem}}$

Any bounded entire function must be a constant.

Proof: Let $|f(z)| \leq K \ \forall z$. From the above corollary $|f'(z_0)| \leq \frac{K}{r}$. Since $f(z)$ is entire, this is true for every r, so we can take r as large as we want. Thus, letting $r \to \infty$, we see that

$0 \leq |f'(z_0)| \leq 0 \Rightarrow f'(z_0) = 0$

$\Rightarrow f'(z) = 0 \ \forall z$ ($\because z_0$ was an arbitrary point)

$\Rightarrow f(z)$ is constant.

Morera's theorem (Converse of Cauchy's integral thm.)

Thm. 3 If $f(z)$ is continuous in a simply connected domain D and if $\oint_C f(z)\,dz = 0$ for every closed path in D, then $f(z)$ is analytic in D.

Proof: From Sect. 14.2, we showed that if $f(z)$ is analytic in D, then

$$F(z) = \int_{z_0}^{z} f(z^*)\,dz^* \quad \text{is analytic in } D$$

& $F'(z) = f(z)$.

But in the proof, we didn't use analyticity of f at all, We only used the fact that f is continuous & the fact that its integral along every closed path in D is zero. Then we concluded that $F(z)$ is analytic.

Now thm. 1 implies that $F'(z) = f(z)$ is also analytic in D. Thus the theorem is proved. ∎

Sect. 15.2 - Power series

- Very important in complex analysis
- Every analytic function can be represented by power series.
- Their sums are analytic functions.

Defn. A power series in powers of $z - z_0$ is a series of the form

$$\boxed{\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots,}$$

where $\underbrace{z}_{\text{(variable)}} \in \mathbb{C}$, $\underbrace{a_i}_{\text{(constants)}} \in \mathbb{C}$, for $0 \leq i < \infty$, called coefficients of the series, and $\underbrace{z_0}_{\text{(constant)}} \in \mathbb{C}$ called the center of the series.

Special case: $$\boxed{\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots}$$