# MA 502 - SEM. II (PART II) - LECTURE 16

## The Argument Principle

- Suppose $f$ is analytic and has a zero of order $m$ at $z=a$. Then $f(z) = (z-a)^m g(z)$, where $g$ is analytic with $g(a) \neq 0$. This implies

$$\boxed{\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}}, \qquad (1)$$

where $g'/g$ is analytic near $z=a$ ($\because g(a) \neq 0$).

- Suppose $f$ has a pole of order $m$ at $z=a$, then $f(z) = (z-a)^{-m} g(z)$, where $g$ is analytic and $g(a) \neq 0$. This gives $\boxed{\frac{f'(z)}{f(z)} = \frac{-m}{z-a} + \frac{g'(z)}{g(z)}}$, where again $g'/g$ is analytic near $z=a$.

$$(2)$$

**Defn.** A <u>meromorphic function</u> $f$ on an open set $G$ is a function defined and analytic in $G$ except for poles.

- Meromorphic functions can be regarded as 'nearly analytic' functions for the discontinuities at its singularities can be removed although the non-differentiability of them cannot be removed. This is because, if $f$ is a meromorphic function on $G$ & $f: G \to \mathbb{C}_\infty$ is defined with $f(z) = \infty$ whenever $z$ is a pole of $f$, then $f$ is continuous from $G$ into $\mathbb{C}_\infty$. (Tutorial problem).

**Argument principle**: Let $f$ be a meromorphic function in $G$ with poles $p_1, p_2, \cdots, p_m$ and zeros $z_1, z_2, \cdots, z_n$ counted according to multiplicity. If $\gamma$ is a closed rect. curve in $G$ with $\gamma \approx 0$ and not passing through $p_1, p_2, \cdots p_m$ and $z_1, z_2, \cdots, z_n$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{j=1}^{m} n(\gamma; p_j) \quad \text{——} (*)$$

Proof: By a repeated application of (1) and (2),

$$\boxed{\frac{f'(z)}{f(z)} = \sum_{k=1}^{n}\frac{1}{z-z_k} - \sum_{j=1}^{m}\frac{1}{z-p_j} + \frac{g'(z)}{g(z)}}, \quad \text{——} (3)$$

where $g$ is analytic and never vanishes on $G$. Then $g'/g$ is analytic in $G$, hence by Cauchy's thm,

$\frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,dz = 0$. Hence from (3),

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n}\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_k} - \sum_{j=1}^{m}\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-p_j} + 0$$

$$= \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{j=1}^{m} n(\gamma; p_j).$$

$\boxtimes$

**Reason for calling it the argument principle**:

- Note that we cannot define a branch of $\log f(z)$, for otherwise, it would be a primitive of $f'(z)/f(z)$, and then $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$ would be zero, contradicting $(*)$.
- However, since no zero or pole of $f$ lies on $\gamma$, there is a disk $B(a;r)$, for each $a \in \{\gamma\}$, such that a

branch of $\log f(z)$ can be defined on $B(a;r)$.

- This is because, we can select $r$ sufficiently small that $f(z) \neq 0$ or $\infty$ in $B(a;r)$. Since the balls form an open cover of $\{\gamma\}$, by Lebesgue's covering lemma (which states that 'If $(X,d)$ is a sequentially compact space and $\mathcal{G}$ is an open cover of $X$, then $\exists$ $\varepsilon > 0 \ni$ if $x \in X$, there is a set $G$ in $\mathcal{G}$ with $B(x;\varepsilon) \subseteq G$) there is an $\varepsilon > 0 \ni \forall a \in \{\gamma\}$, we can define a branch of $\log f(z)$ on $B(a;\varepsilon)$.

- Now suppose $\gamma$ is defined on $[0,1]$. Being uniformly continuous, $\exists$ a partition $0 = t_0 < t_1 < \dots < t_k = 1$ such that $\gamma(t) \in B(\gamma(t_{j-1});\varepsilon)$ for $t_{j-1} \leq t \leq t_j$ and $1 \leq j \leq k$.

Let $l_j$ be a branch of $\log f$ defined on $B(\gamma(t_{j-1});\varepsilon)$ for $1 \leq j \leq k$.

Since $\gamma(t_j)$ lies in both $j$th and $(j+1)$-st ball, we can choose $l_1, \dots, l_k$ so that

$$l_1(\gamma(t_1)) = l_2(\gamma(t_1)) \quad ; \quad l_2(\gamma(t_2)) = l_3(\gamma(t_2)) \quad ; \dots ;$$
$$l_{k-1}(\gamma(t_{k-1})) = l_k(\gamma(t_{k-1})).$$

- Suppose $\gamma_j$ is the path restricted to $[t_{j-1}, t_j]$, then $l_j' = f'/f$ implies

$$\int_{\gamma_j} \frac{f'}{f} = l_j(\gamma(t_j)) - l_j(\gamma(t_{j-1})) \quad \text{for } 1 \leq j \leq k.$$

Summing both sides from $j = 1$ to $k$, we have

$$\boxed{\int_\gamma \frac{f'}{f} = l_k(a) - l_1(a)} \quad \text{with } a = \gamma(0) = \gamma(1)$$

Using the above theorem,

$$l_k(a) - l_1(a) = 2\pi i K, \quad \text{where } K = \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{j=1}^{m} n(\gamma; p_j)$$

Because $2\pi i K$ is purely imaginary,

$$\operatorname{Im}(l_k(a)) - \operatorname{Im}(l_1(a)) = 2\pi K$$

. Note that $\operatorname{Im}(l(a)) = \arg f(a)$.

Thus, as $z$ traces out $\gamma$, $\arg f(z)$ changes by $2\pi K$. Hence the name, argument principle.

**Thm.** Let $f$ be meromorphic in the region $G$ with zeros $z_1, z_2, \dots, z_n$ and poles $p_1, p_2, \dots, p_m$ counted according to multiplicity. If $g$ is analytic in $G$ & $\gamma$ is a closed rectifiable curve in $G$ with $\gamma \approx 0$ and not passing through any $z_i$ or $p_j$, then

$$\frac{1}{2\pi i}\int_\gamma g(z)\frac{f'(z)}{f(z)}\,dz = \sum_{i=1}^{n} g(z_i)\, n(\gamma; z_i) - \sum_{j=1}^{m} g(p_j)\, n(\gamma; p_j)$$

Proof :- Multiply (3) by $g$ and then integrate. $\blacksquare$

**Formula for calculating an inverse of a 1-1 analytic fn.**

We have previously shown that if $f$ is analytic & 1-1, then $f^{-1}$ is analytic. The above theorem can be used to give an explicit formula for calculating this inverse.

Suppose $R > 0$ and $f$ is analytic in $\bar{B}(a;R)$ & also 1-1 there. Let $\Omega = f(B(a;R))$. If $|z-a| < R$ & $\xi = f(z) \in \Omega$, then $f(w) - \xi$ has one and only one zero in $B(a;R)$ (see Tutorial prob.). If $g(w) = w$, then by the above thm;

$$z = \frac{1}{2\pi i} \int_\gamma \frac{w f'(w)}{f(w) - \xi} \, dw ,$$

where $\gamma$ is the circle $|w-a| = R$. But $z = f^{-1}(\xi)$. Hence,

Thm. Let $f$ be analytic on an open set containing $\bar{B}(a;R)$ and suppose that $f$ is 1-1 on $B(a;R)$. If $\Omega = f(B(a;R))$ and $\gamma$ is the circle $|z-a| = R$, then $f^{-1}(\omega)$ is defined for each $\omega$ in $\Omega$ by

$$f^{-1}(\omega) = \frac{1}{2\pi i} \int_\gamma \frac{z f'(z) \, dz}{f(z) - \omega} .$$

ROUCHÉ'S THEOREM

Suppose $f$ and $g$ are meromorphic in a nbhd $\bar{B}(a;R)$ of $a$ with no zeros or poles on $\gamma : |z-a| = R$. If $Z_f, Z_g$ $(P_f, P_g)$ denote the number of zeros (poles) of $f$ and $g$ inside $\gamma$ counted according to multiplicities and if

$$|f(z) + g(z)| < |f(z)| + |g(z)| \text{ on } \gamma, \text{ then}$$

$$Z_f - P_f = Z_g - P_g .$$

Proof: Note that $\left| \frac{f(z)}{g(z)} + 1 \right| < \left| \frac{f(z)}{g(z)} \right| + 1$.

If $\lambda = \frac{f(z)}{g(z)}$ and $\lambda \in \mathbb{R}^+$, then $\lambda + 1 < \lambda + 1$, a contradiction.

Hence the meromorphic function $f/g$ maps $\gamma$ onto $\Omega = \mathbb{C} - [0, \infty)$. So we can define a branch of logarithm on $\Omega$. Thus $\ell(f(z)/g(z))$ is a well-defined primitive of $(f/g)'/(f/g)$ in a nbhd of $\gamma$ whence

$$0 = \frac{1}{2\pi i}\int_\gamma (f/g)'/(f/g) = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} - \frac{1}{2\pi i}\int_\gamma \frac{g'}{g}$$

$$= (Z_f - P_f) - (Z_g - P_g).$$

$\boxtimes$