15.3 – Functions given by power series

- If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has a non-zero radius of convergence $(R>0)$, its sum is a function of z, $f(z)$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots \qquad (|z|<R).$$

$f(z)$ is represented by a power series or developed in the power series.

e.g. $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z|<1$.

- Uniqueness of a power series representation

– A function cannot be represented by two different powerseries with the same center.

- Continuity of the sum of a power series

If a function $f(z)$ can be represented by $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R>0$, then $f(z)$ is continuous at $z=0$.

Thm. (Identity theorem for power series)

Let the power series $a_0+a_1z+a_2z^2+\ldots$ and $b_0+b_1z+b_2z^2+\ldots$ both be convergent for $|z|<R$, $R>0$, and let both have the same sum $\forall z$. Then the series are identical, that is, $a_0=b_0$, $a_1=b_1$, $a_2=b_2$, ...

Hence if a function $f(z)$ can be represented by a power series with any center z_0, this representation is unique.

Operations on power series

Term wise addition or subtraction of two power series with radii of convergence R_1 & R_2 yields a power series with radius of convergence $\min(R_1, R_2)$.

Term wise multiplication

(Multiplication of each term of one series with each term of the other.) — CAUCHY PRODUCT

$$\left(\sum_{k=0}^{\infty} a_k z^k\right)\left(\sum_{m=0}^{\infty} b_m z^m\right)$$

$$= (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots)$$

$$= a_0 b_0 + (a_1 b_0 + a_0 b_1) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots$$

$$= \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n .$$

This power series converges absolutely for each z within the smaller circle of convergence of the two given series & has the sum $f(z) g(z)$, where

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \& \quad g(z) = \sum_{m=0}^{\infty} b_m z^m .$$

- **Term wise differentiation** of a power series is permissible. If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + \cdots$, then $f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \cdots$ is called the derived series of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$.

Thm. The derived series of a power series has the same radius of convergence as the original series.

Termwise integration of power series

The power series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} = a_0 z + \frac{a_1}{2} z^2 + \frac{a_2}{3} z^3 + \cdots$ obtained by integrating the series $a_0 + a_1 z + a_2 z^2 + \cdots$ term by term has the same radius of convergence as the original series.

Thm. (Power series represent analytic functions)
A power series with a non-zero radius of convergence R represents an analytic function at every point interior to its circle of convergence. The derivatives of this function are obtained by differentiating the original series term by term. All the series thus obtained have the same radius of convergence as the original series. Hence each of them represents an analytic function.

Example: $\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n} (z-2i)^n$

① Direct way: $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n(n-1)}{2^n} \frac{2^{n+1}}{(n+1)n}$.

$$= 2 \lim_{n\to\infty} \left(\frac{1-1/n}{1+1/n}\right) = 2 .$$

$$= \frac{(z-2i)^2}{2^2} \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n-2}} (z-2i)^{n-2}$$

② Note that

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n-2}}(z-2i)^{n-2} = \sum_{n=2}^{\infty} n(n-1)\left(\frac{z}{2}-i\right)^{n-2}$$

$$= \sum_{n=2}^{\infty} n(n-1)(w-i)^{n-2}, \text{ where } w=\frac{z}{2}.$$

Now $\displaystyle\sum_{n=2}^{\infty} n(n-1)(w-i)^{n-2} = \frac{d^2}{dw^2}\sum_{n=0}^{\infty}(w-i)^n$.

Since $\displaystyle\sum_{n=0}^{\infty}(w-i)^n$ has radius of convergence 1, that is, it converges for all w with $|w-i|<1$.

$\Rightarrow |i-\frac{z}{2}|<1 \Rightarrow |z-2i|<2$.

Hence for the series $\displaystyle\sum_{n=2}^{\infty}\frac{n(n-1)}{2^n}(z-2i)^n$, the radius of convergence is 2.

15.4 - Taylor and Maclaurin Series

- So far we have seen that power series represent analytic functions in some open disk.
- Here we will see that any analytic function can be represented by a power series called Taylor series.

Defn- The Taylor series of a function $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \text{ where } a_n = \frac{f^{(n)}(z_0)}{n!}, \text{ or,}$$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}\,dz^*$$

(f analytic on & everywhere inside C)

C ↑ simple closed path containing z_0 in its interior. (clockwise)

The remainder of $f(z) = \sum_{n=0} a_n (z-z_0)^n$ after the n^{th} term is

$$\boxed{R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)\, dz}{(z^*-z_0)^{n+1}(z^*-z)}.} \longrightarrow (*)$$

$\Rightarrow$ TAYLOR'S FORMULA (with remainder)

$$f(z) = f(z_0) + f'(z_0) \cdot \frac{(z-z_0)}{1!} + \frac{f''(z_0)\,(z-z_0)^2}{2!} + \dots + \frac{(z-z_0)^n}{n!} f^{(n)}(z_0) + R_n(z).$$

- Maclaurin series is Taylor series with $z_0 = 0$.

THM. 1 (TAYLOR'S THEOREM)

Let $f(z)$ be analytic in a domain D, and let $z = z_0$ be any point in D. Then there exists precisely one Taylor's series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ with $a_n = \frac{f^{(n)}(z_0)}{n!}$ that represents $f(z)$. This representation is valid in the largest open disk with center z_0 in which $f(z)$ is analytic. The remainder $R_n(z)$ is given by $(*)$. The coefficients a_n satisfy the inequality

$$|a_n| \le \frac{M}{r^n},$$

where M is the maximum of $|f(z)|$ on a circle $|z-z_0| = r$ in D whose interior is also in D.

THM. 2 A power series with non-zero radius of convergence is the Taylor series of its sum.

Proof:- $f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \dots$

Note that $f(z_0) = a_0$.

Also, $f'(z) = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \dots \Rightarrow f'(z_0) = a_1$

$f''(z) = 2a_2 + 6a_3(z-z_0) + \dots \Rightarrow 2!\,a_2 = f''(z_0).$

In general, $f^{(n)}(z_0) = n!\, a_n$.