$$z = \frac{1}{2\pi i}\int_\gamma \frac{w f'(w)}{f(w)-\xi}\,dw,$$

where γ is the circle $|w-a|=R$. But $z=f^{-1}(\xi)$. Hence,

Thm. Let f be analytic on an open set containing $\bar{B}(a;R)$ and suppose that f is 1-1 on $B(a;R)$. If $\Omega = f(B(a;R))$ and γ is the circle $|z-a|=R$, then $f^{-1}(\omega)$ is defined for each ω in Ω by

$$f^{-1}(\omega) = \frac{1}{2\pi i}\int_\gamma \frac{z f'(z)\,dz}{f(z)-w}.$$

ROUCHE'S THEOREM (VERSION I)

Suppose f and g are meromorphic in a nbhd $\bar{B}(a;R)$ of a with no zeros or poles on $\gamma: |z-a|=R$. If Z_f, Z_g (P_f, P_g) denote the number of zeros (poles) of f and g inside γ counted according to multiplicities and if

$$|f(z)+g(z)| < |f(z)|+|g(z)| \text{ on } \gamma, \text{ then}$$

$$Z_f - P_f = Z_g - P_g.$$

Proof: Note that $\left|\frac{f(z)}{g(z)}+1\right| < \left|\frac{f(z)}{g(z)}\right|+1.$

If $\lambda = \frac{f(z)}{g(z)}$ and $\lambda \in \mathbb{R}^+$, then $\lambda+1 < \lambda+1$, a contradiction. Hence the meromorphic function f/g maps γ onto $\Omega = \mathbb{C} - [0,\infty)$. So we can define a branch of logarithm on Ω. Thus $\ell(f(z)/g(z))$ is a well-defined primitive of $(f/g)'/(f/g)$ in a nbhd of γ whence

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{(f/g)'}{(f/g)} = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} - \frac{1}{2\pi i}\int_\gamma \frac{g'}{g}$$

$$= (Z_f - P_f) - (Z_g - P_g).$$

$\boxtimes$

ROUCHÉ'S THEOREM (VERSION II)

Suppose f and g are meromorphic in a nbhd of $\bar{B}(a;R)$ of a with no zeros or poles on $\gamma : |z-a|=R$. If Z_f, Z_g (P_f, P_g) denote the number of zeros (poles) of f and g inside γ counted according to multiplicities and if $0<|g(z)|<|f(z)|<\infty$ on γ, then

$$Z_{f+g} - P_{f+g} = Z_f - P_f.$$

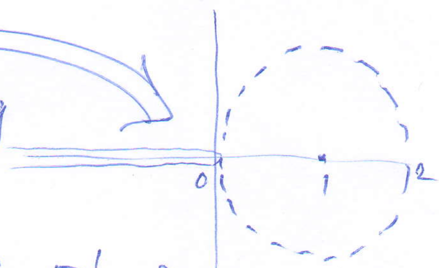
Proof: Note that the condition $0<|g(z)|<|f(z)|<\infty$ implies that f and g do not have any poles or zeros on γ.

Let $F(z) := \dfrac{f(z)+g(z)}{f(z)}$. Then,

$$|F(z)-1| = \left|\frac{f(z)+g(z)}{f(z)} - 1\right| = \left|\frac{g(z)}{f(z)}\right| < 1 \text{ on } \gamma.$$

Let $w = F(z)$. Then $|w-1|<1$

Thus we can define a branch of log (actually a principal branch)



$\Rightarrow \log F(z)$ is the primitive of $\dfrac{F'(z)}{F(z)}$ on γ.

But $$\frac{F'(z)}{F(z)} = \frac{\left(\frac{f+g}{f}\right)'}{\left((f+g)/f\right)} = \frac{[f(f+g)' - (f+g)f']}{f^2} \cdot \frac{f}{f+g}$$

$$= \frac{(f+g)'}{f+g} - \frac{f'}{f}.$$

$$\Rightarrow 0 = \frac{1}{2\pi i}\int_\gamma \frac{F'(z)}{F(z)} = \frac{1}{2\pi i}\int_\gamma \frac{(f+g)'}{f+g} - \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}$$

$$\Rightarrow Z_{f+g} - P_{f+g} = Z_f - P_f.$$

$\boxtimes$

ROUCHÉ'S THEOREM (VERSION III)

Same hypotheses as before, but $|f+g| < |g|$ on γ. Then $Z_f - P_f = Z_g - P_g$.

Proof:- Replace $g \to f+g$ & f by $-g$ in Version II of Rouché's theorem. Then if $0 < |f+g| < |g| < \infty$, we have

$$Z_{-g+f+g} - P_{-g+f+g} = Z_{-g} - P_{-g}$$

However note that g and $-g$ have same poles and zeros.

$$\Rightarrow Z_f - Z_g = Z_g - P_g.$$

$\boxtimes$

A new proof of the Fundamental Theorem of Algebra

Let $p(z) := z^n + a_1 z^{n-1} + \dots + a_n$. Then

$$\frac{p(z)}{z^n} = 1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots + \frac{a_n}{z^n}$$

$\lim_{z \to \infty} \frac{p(z)}{z^n} = 1$ implies that there is a sufficiently large R $\ni$ $\left| \frac{p(z)}{z^n} - 1 \right| < 1$ for $|z| = R$, i.e.,

$$|\underset{\uparrow \atop f}{p(z)} - \underset{\uparrow \atop g}{z^n}| < |z|^n = |\underset{\uparrow \atop g}{-z^n}| \text{ on } |z| = R$$

Hence by version III of Rouche's thm,

$$Z_f = Z_g.$$

But $-z^n$ has n zeros inside $|z| = R$.

Hence $p(z)$ has n zeros inside $|z| = R$.