$f(z)$ can be represented by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

consisting of non-negative powers & the 'principal part' (negative powers). The coefficients are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^* \quad \& \quad b_n = \frac{1}{2\pi i} \oint_C (z^*-z_0)^{n-1} f(z^*)\, dz^*,$$

where C (simple closed path) is traversed counterclockwise, ~~arou~~ lies in the annulus & encircles the inner circle.

Remark: We can also write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \text{ where}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)\, dz^*}{(z^*-z_0)^{n+1}}, \quad (n \in \mathbb{Z}).$$

Examples Find the Laurent series of the following functions (with centre 0):

① $$\frac{\cos z}{z^4} = \frac{1}{z^4} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \frac{1}{z^4}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)$$

$$= \frac{1}{z^4} - \frac{1}{2z^2} + \frac{1}{24} - \frac{z^2}{720} + \cdots \quad (\text{for } |z| > 0).$$

② $$\frac{1}{z(z-1)} = \frac{z-(z-1)}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}$$

Now if $|z| < 1$, $\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n$

$$\Rightarrow \frac{1}{z(z-1)} = \frac{-1}{z} - \sum_{n=0}^{\infty} z^n, \text{ if } 0 < |z| < 1.$$

If $|z|>1$, then we write

$$\frac{1}{z-1} = \frac{1/z}{1-1/z} = \sum_{n=1}^{\infty}\left(\frac{1}{z}\right)^n \text{, since } |1/z|<1 .$$

Hence
$$\frac{1}{z(z-1)} = -\frac{1}{z} + \sum_{n=1}^{\infty}\frac{1}{z^n} = \sum_{n=2}^{\infty}\frac{1}{z^n} \text{. for } |z|>1.$$

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*Develop $f(z) = \dfrac{2z-3i}{z^2-3iz-2}$ in a series valid for

(1) $0<|z|<1$ (2) $1<|z|<2$ (3) $|z|>2$

(4) $0<|z+i|<2$.

Note that
$$f(z)= \frac{2z-3i}{z^2-3iz-2} = \frac{(z-2i)+(z-i)}{(z-2i)(z-i)}$$
$$= \frac{1}{z-i} + \frac{1}{z-2i}.$$

(1)
$$\frac{1}{z-i} = \cancel{\frac{1}{-i(1+z)}}\ \frac{1}{-i\left(1+\frac{z}{-i}\right)} = \frac{i}{1+iz}$$
$$= i\sum_{n=0}^{\infty}(-iz)^n \quad (\because |-iz| = |z| < 1)$$

Also
$$\frac{1}{z-2i} = \frac{1}{-2i\left(1+\frac{z}{-2i}\right)} = \frac{i}{2\left(1+\frac{iz}{2}\right)} = \frac{i}{2}\sum_{n=0}^{\infty}\left(-\frac{iz}{2}\right)^n$$
$$\left(\because \left|\frac{iz}{2}\right| = \frac{|z|}{2} < \frac{1}{2} < 1\right)$$

Hence
$$f(z) = i\sum_{n=0}^{\infty}(-iz)^n + \frac{i}{2}\sum_{n=0}^{\infty}\left(-\frac{iz}{2}\right)^n \quad (\text{for } |z| < 1)$$

(2)

$$\frac{1}{z-i} = \frac{1}{z\left(1-\frac{i}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n, \text{ since } \left|\frac{i}{z}\right| = \left|\frac{1}{z}\right| < 1. \quad \text{—— (c)}$$

Thus from (b) & (c),

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n - \frac{1}{2i}\sum_{n=0}^{\infty}\left(-\frac{iz}{2}\right)^n \quad \text{for } 1<|z|<2.$$

(3) $\boxed{|z|>2}$

Note that $\frac{1}{z-2i} = \frac{1}{z\left(1-\frac{2i}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2i}{z}\right)^n$,

since $\left|\frac{2i}{z}\right| = \frac{2}{|z|} < 1.$ —— (d)

Thus from (c) & (d), when $|z|>2$, we have

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2i}{z}\right)^n$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\frac{(1+2^n)i^n}{z^n}.$$

$$= \frac{2}{z} + \frac{3i}{z^2} - \frac{5}{z^3} - \frac{9i}{z^4} + \dots$$

(4) $\boxed{0<|z+i|<2}$

$$\frac{1}{z-i} = \frac{1}{z+i-2i} = \frac{1}{-2i\left(1-\frac{z+i}{2i}\right)} = \frac{i}{2}\sum_{n=0}^{\infty}\left(\frac{z+i}{2i}\right)^n,$$

since $\left|\frac{z+i}{2i}\right| = \frac{|z+i|}{2} < 1.$

$$\frac{1}{z-2i} = \frac{1}{z+i-3i} = \frac{1}{-3i\left(1-\frac{z+i}{3i}\right)} = \frac{i}{3}\sum_{n=0}^{\infty}\left(\frac{z+i}{3i}\right)^n$$

Since $\left|\frac{z+i}{3i}\right| = \frac{|z+i|}{3} < 1$, which is true since $|z+i|<2$.

$$\Rightarrow \boxed{f(z) = \frac{i}{2}\sum_{n=0}^{\infty}\left(\frac{z+i}{2i}\right)^n + \frac{i}{3}\sum_{n=0}^{\infty}\left(\frac{z+i}{3i}\right)^n}$$

Sect. 16.2 – Singularities & zeros

- A singular point of an analytic function $f(z)$ is a point at which $f(z)$ ceases to be analytic.
- A zero of a function at a point 'a' is that point for which $f(a) = 0$.

Precise definition of singularity

- A function $f(z)$ is singular or has a singularity at a point $z = z_0$ if $f(z)$ is not analytic (perhaps not even defined) at $z = z_0$, but every neighborhood of $z = z_0$ contains points at which $f(z)$ is analytic.

ISOLATED SINGULARITY

- $z = z_0$ is an isolated singularity of $f(z)$ if $z = z_0$ has a neighborhood without further singularities of $f(z)$.

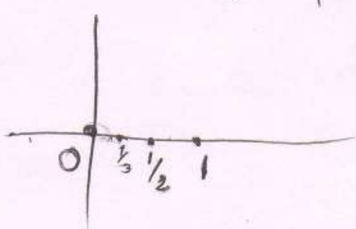
e.g. ① $f(z) = \dfrac{1}{(z-3)(z+5)}$ has 2 isolated singularities at $z = 3$ & $z = -5$.

② $\tan z$ has isolated singularities at $\pm \pi/2, \pm 3\pi/2, \ldots$

NON-ISOLATED SINGULARITY

- The function $\dfrac{1}{\sin(\frac{\pi}{z})}$ has a non-isolated singularity at $z = 0$:

① singularities of the function are the zeros of $\sin(\frac{\pi}{z})$, i.e. the points $z = 1/n$, $n \in \mathbb{N}$.



- Isolated singularities of $f(z)$ at $z=z_0$ can be classified by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n},$$

which is valid in $\underline{0<|z-z_0|<R}$. (immediate neighborhood of $z=z_0$)

This is done as follows:

(i) If the principal part $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ has only finitely many terms, that is,

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}, \quad b_i \neq 0 \ (1 \le i \le m),$$

then the singularity of $f(z)$ at $z=z_0$ is called a pole, and m is called its order.

- Poles of order 1 are called <u>simple poles.</u>

(ii) If the principal part has infinitely many terms, then we say that $f(z)$ has at $z=z_0$ an isolated <u>essential singularity</u>.

(iii) If the principal part does not contain any term, i.e. $b_1 = b_2 = b_3 = \cdots = 0$, then $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ is said to have a removable singularity, and the function $f(z)$ is actually analytic at $z=z_0$.

e.g.:- $$\boxed{\frac{\sin z}{z}} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

- If f has an isolated singularity at a, then $z=a$ is a removable singularity if & only if $\lim_{z\to a} (z-a) f(z) = 0$.

Examples of poles & isolated essential singularities:

① Poles:

$$f(z) = \sin(z+1) + \frac{1}{z(z+2)^3} + \frac{5}{(z+2)^2}$$

has a simple pole at $z=0$ and pole of order 3 at $z=-2$.

② $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\,z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$

and $\cos\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{-2n}}{(2n)!} = 1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \cdots$

have essential singularities at $z=0$.

Another example:

① $\frac{1}{z^3-z^4} = \frac{-1}{z^4(1-\frac{1}{z})} = \frac{-1}{z^4}\sum_{n=0}^{\infty}\frac{1}{z^n}$ for $|z|>1$

$= -\frac{1}{z^4}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$.

Does this mean $\frac{1}{z^3-z^4}$ has essential singularity at $z=0$?

Ans. No! We need immediate neighborhood!

Thm. (Poles) If $f(z)$ is analytic and has a pole at $z=z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ in any manner.

e.g. $\frac{1}{z^2}$ has a pole at $z=0$, & $\lim_{z\to 0}\left|\frac{1}{z^2}\right| = \infty$

Behavior near an essential singularity

- $e^{1/z}$ has an essential singularity at $z=0$

(i) If z tends to zero along imaginary axis, that is, $z = iy$, with $y \to 0$, then $e^{1/z} = e^{1/(iy)} = e^{-i/y}$ does not approach any limit as $y \to 0$.

(ii) If $z \to 0$ through positive real values, $e^{1/z} \to \infty$

(iii) If $z \to 0$ through negative real values, $e^{1/z} \to 0$.

In fact, $e^{1/z}$ takes any given value $c = c_0 e^{i\alpha} \neq 0$ in an arbitrarily small neighborhood of $z=0$:

Proof/ Let $z = re^{i\theta}$. Then we have to solve the eqn.

$$e^{1/z} = e^{(\cos\theta - i\sin\theta)/r} = c_0 e^{i\alpha}. \text{ for } r \ \& \ \theta.$$

$$\Rightarrow e^{\frac{\cos\theta}{r} - i\frac{\sin\theta}{r}} = e^{\ln c_0 + i\alpha}$$

$$\Rightarrow \cos\theta = r\ln(c_0) \ \& \ -\sin\theta = \alpha r.$$

Thus $\cos^2\theta + \sin^2\theta = 1$ implies

$$r^2\ln^2(c_0) + \alpha^2 r^2 = 1 \Rightarrow r^2 = \frac{1}{(\ln c_0)^2 + \alpha^2}$$

$$\& \ \tan\theta = -\frac{\alpha}{\ln c_0}.$$

Thus r can be made arbitrarily small by adding multiples of 2π to α, without changing the value of c.

Thm. (Picard's theorem)

If $f(z)$ is analytic and has an isolated essential singularity at a point z_0, it takes on every complex value, with at most one exception, in an arbitrarily small neighborhood of z_0.