

$f(z)$ can be represented by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

consisting of non-negative powers & the 'principal part' (negative powers). The coefficients are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^* \quad \& \quad b_n = \frac{1}{2\pi i} \oint_C (z^*-z_0)^{n-1} f(z^*) dz^*$$

where C is traversed counterclockwise, ~~around~~ lies in the annulus & encircles the inner circle.
(simple closed path)

Remark: We can also write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \text{ where}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^*, \quad (n \in \mathbb{Z})$$

Examples

Find the Laurent series of the following functions (with center 0):

$$\begin{aligned} \textcircled{1} \frac{\cos z}{z^4} &= \frac{1}{z^4} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \frac{1}{z^4} \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \right) \\ &= \frac{1}{z^4} - \frac{1}{2z^2} + \frac{1}{24} - \frac{z^2}{720} + \dots \quad (\text{for } |z| > 0) \end{aligned}$$

$$\textcircled{2} \frac{1}{z(z-1)} = \frac{z-(z-1)}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}$$

Now if $|z| < 1$, $\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n$

$$\Rightarrow \frac{1}{z(z-1)} = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n, \text{ if } 0 < |z| < 1$$

$$= \frac{1}{1} + \frac{1}{1}$$

Note that $f(z) = \frac{z^2 - 3!}{(z - 2!)(z - 1!)} = \frac{(z - 2!)(z - 1!)}{(z - 2!)(z - 1!)} = \frac{(z - 2!)(z - 1!)}{(z - 2!)(z - 1!)} + \frac{(z - 2!)(z - 1!)}{(z - 2!)(z - 1!)}$

④ $0 < |z| + 1 < 2$

① $0 < |z| < 1$

② $1 < |z| < 2$

③ $|z| > 2$

Develop $f(z) = \frac{z^2 - 3!}{z^2 - 3!z - 2}$ in a series valid for

~~in a series~~

- A singular point of an analytic function $f(z)$ is a point at which $f(z)$ ceases to be analytic.
- A zero of a function at a point 'a' is that point for which $f(a) = 0$.

Precise definition of singularity

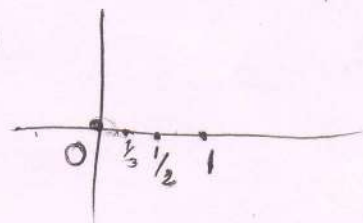
- A function $f(z)$ is singular or has a singularity at a point $z = z_0$ if $f(z)$ is not analytic (perhaps not even defined) at $z = z_0$, but every neighborhood of $z = z_0$ contains points at which $f(z)$ is analytic.

ISOLATED SINGULARITY

- $z = z_0$ is an isolated singularity of $f(z)$ if $z = z_0$ has a neighborhood without further singularities of $f(z)$.
- e.g. ① $f(z) = \frac{1}{(z-3)(z+5)}$ has 2 isolated singularities at $z = 3$ & $z = -5$.
- ② $\tan z$ has isolated singularities at $\pm \pi/2, \pm 3\pi/2, \dots$

NON-ISOLATED SINGULARITY

- The function $\sin\left(\frac{\pi}{z}\right)$ has a non-isolated singularity at $z = 0$:
- ① singularities of the function are the zeros of $\sin\left(\frac{\pi}{z}\right)$, i.e. the points $z = 1/n, n \in \mathbb{N}$.



• Isolated singularities of $f(z)$ at $z=z_0$ can be classified by the Laurent Series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n},$$

which is valid in $0 < |z-z_0| < R$. (immediate neighborhood of $z=z_0$)

This is done as follows:

(i) If the principal part $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ has only finitely many ~~poles~~ terms, that is,

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_m}{(z-z_0)^m}, \quad b_i \neq 0, \quad (1 \leq i \leq m),$$

then the singularity of $f(z)$ at $z=z_0$ is called a pole, and m is called its order.

• Poles of order 1 are called simple poles.

(ii) If the principal part has infinitely many terms, then we say that $f(z)$ has at $z=z_0$ an isolated essential singularity.

(iii) If the principal part does not contain any term, i.e. $b_1 = b_2 = b_3 = \dots = 0$, then $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is said to have a removable singularity, and the function $f(z)$ is actually analytic at $z=z_0$.

e.g. :- $\boxed{\frac{\sin z}{z}} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$

If f has an isolated singularity at a , then $z=a$ is a removable singularity if & only if $\lim_{z \rightarrow a} (z-a)f(z) = 0$.

Examples of poles & isolated essential singularities 185

① Poles:

$$f(z) = \sin(z+1) + \frac{1}{z(z+2)^3} + \frac{5}{(z+2)^2}$$

has a simple pole at $z=0$ and pole of order ~~2~~ ³ at $z=-2$,

② $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$

and $\cos \frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{-2n}}{(2n)!} = 1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \dots$

have essential singularities at $z=0$.

Another example:

① $\frac{1}{z^3 - z^4} = \frac{-1}{z^4(1 - \frac{1}{z})} = -\frac{1}{z^4} \sum_{n=0}^{\infty} \frac{1}{z^n} \quad \text{for } |z| > 1$

$$= -\frac{1}{z^4} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right).$$

Does this mean $\frac{1}{z^3 - z^4}$ has ~~a~~ essential singularity at $z=0$?

Ans. No! We need immediate neighborhood!

Thm. (Poles) If $f(z)$ is analytic and has a pole at $z=z_0$, then $|f(z)| \rightarrow \infty$ as $z \rightarrow z_0$ in any manner.

e.g. $\frac{1}{z^2}$ has a pole at $z=0$, & $\lim_{z \rightarrow 0} \left| \frac{1}{z^2} \right| = \infty$

Behavior near an essential singularity

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- $e^{1/z}$ has an essential singularity at $z=0$
 - (i) If z tends to zero along imaginary axis, that is, $z = iy$, with $y \rightarrow 0$, then $e^{1/z} = e^{1/(iy)} = e^{-i/y}$ does not approach any limit as $y \rightarrow 0$.
 - (ii) If $z \rightarrow 0$ through positive real values, $e^{1/z} \rightarrow \infty$
 - (iii) If $z \rightarrow 0$ through negative real values, $e^{1/z} \rightarrow 0$.

In fact, $e^{1/z}$ takes any given value $C = c_0 e^{i\alpha} \neq 0$ in an arbitrarily small neighborhood of $z=0$;

Proof
Let $z = r e^{i\theta}$. Then we have to solve the eqn.

$$e^{1/z} = \frac{e^{\cos\theta - i\sin\theta}}{r} = c_0 e^{i\alpha}, \text{ for } r \text{ \& } \theta.$$

$$\Rightarrow e^{\frac{\cos\theta - i\sin\theta}{r}} = e^{\ln c_0 + i\alpha}$$

$$\Rightarrow \cos\theta = r \ln(c_0) \quad \& \quad -\sin\theta = \alpha r.$$

Thus $\cos^2\theta + \sin^2\theta = 1$ implies

$$r^2 \ln^2(c_0) + \alpha^2 r^2 = 1 \Rightarrow r^2 = \frac{1}{(\ln c_0)^2 + \alpha^2}$$

$$\& \quad \tan\theta = -\frac{\alpha}{\ln c_0}.$$

Thus r can be made arbitrarily small by adding multiples of 2π to α , without changing the value of c .

Thm. (Picard's theorem)

If $f(z)$ is analytic and has an isolated essential singularity at a point z_0 , it takes on every complex value, with at most one exception, in an arbitrarily small neighborhood of z_0 .