

Zeros of analytic functions

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A zero of an analytic function f has order n if not only f is zero at that point, but also $f', f'', \dots, f^{(n-1)}$ are zero at $z = z_0$, and $f^{(n)}(z_0) \neq 0$.

• A first order zero is called a simple zero.

Eg. (i) $(z+a)^4$ has a fourth-order zero at $z = -a$.

(ii) $1+z^2$ has simple zeros at $\pm i$.

(iii) e^z has no zeros.

(iv) $1 - \cos z$ has second-order zeros at $0, \pm 2\pi, \pm 4\pi, \dots$

Taylor series at a zero:

If f has an m^{th} -order zero at $z = z_0$, then its Taylor series is given by

$$f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n, \quad (a_m \neq 0)$$

This is characteristic of such a zero, because if $f(z) = \sum_{n=m}^{\infty} a_n (z-z_0)^n$, by differentiation, it follows

that f has an m^{th} -order zero.

Thm. (Zeros)

The zeros of an analytic function $f(z) (\neq 0)$ are isolated; that is, each of them has a neighborhood that contains no further zeros of $f(z)$.

Proof: If f has an m^{th} order zero at $z = z_0$, then $f(z) = (z-z_0)^m (a_m + a_{m+1}(z-z_0) + \dots)$, where $a_m \neq 0$.

Note that $(z-z_0)^m$ is zero only at $z=z_0$.
 The power series $a_m + a_{m+1}(z-z_0) + \dots$ represents an analytic function, say $g(z)$. Note that $g(z_0) = a_m \neq 0$. Since an analytic function is continuous, and hence $g(z) \neq 0$ in some neighborhood of $z=z_0$, also, so is true with $f(z)$. ▣

Thm. (Poles & zeros)

Let $f(z)$ be analytic at $z=z_0$ and have a zero of n^{th} order at $z=z_0$. Then $1/f(z)$ has a pole of n^{th} order at $z=z_0$.

The same holds for $h(z)/f(z)$ if $h(z)$ is analytic at $z=z_0$ & $h(z_0) \neq 0$.

Analyticity / Singularity at infinity

If we want to investigate $f(z)$ for large $|z|$, set $z=1/w$ and investigate $f(z) = f(1/w) = g(w)$ in a neighborhood of $w=0$.

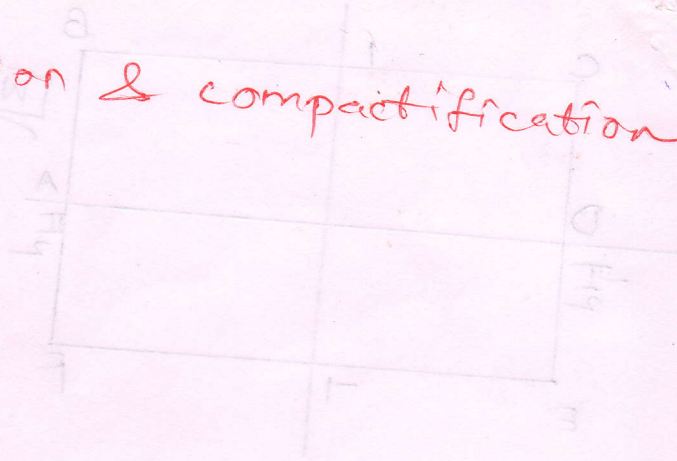
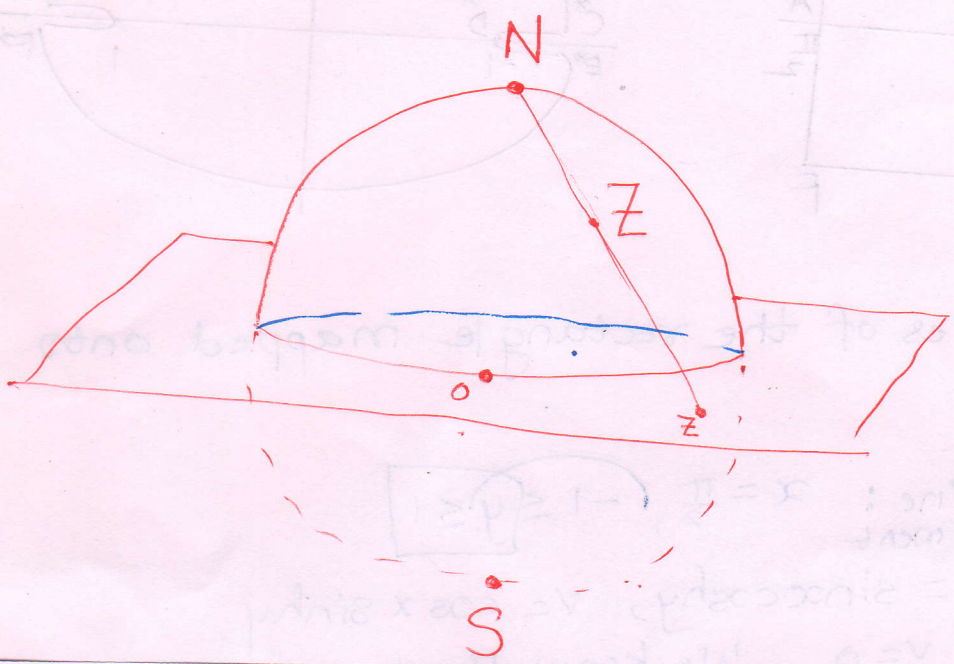
$f(z)$ is defined as analytic or singular at infinity if $g(w)$ is analytic or singular, ~~at~~ resp; at $w=0$.

If $\lim_{w \rightarrow 0} g(w)$ exists, we define $g(0) = \lim_{w \rightarrow 0} g(w)$.

• We say $f(z)$ has an n^{th} order zero at infinity if $f(1/w)$ has such a zero at $w=0$. Similarly for poles and essential singularities.

Eg. $f(z) = 1/z^2$ analytic at $\infty \Leftrightarrow f(1/z) = z^2$ analytic at 0.
 $f(z) = z^3$ singular at ∞ .
 \Updownarrow
 $f(1/z) = 1/z^3$ singular at 0.

Stereographic projection & compactification



Upper & lower sides of the sphere mapped onto semi-ellipses.

At the vertical line: $x = 1$ segment
Note that $u = \sin \theta$ and $v = \cos \theta$
 $\Rightarrow u = \cos \theta, v = \sin \theta$