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Review of complex numbers, complex plane, Polar forms, powers and roots

- A complex number is an ordered pair (x, y) of real numbers x and y, written $z = (x, y)$.

  x: Real part of z; y: Imag. part of z.

- $(x, y) \neq (y, x)$ unless $x = y$.
- $(x, y) = (a, b)$ if and only if $x = a$ & $y = b$.
  (iff)
- Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Then define

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$$

& $$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).$$

- These definitions of addition and multiplication turn the set of all complex numbers into a field with $(0,0)$ & $(1,0)$ taking the role of 0 & 1 in the set of real numbers

- $(x, 0) + (y, 0) = (x + y, 0)$
- $(x, 0)(y, 0) = (xy, 0)$

as in the case of addition & multiplication of real numbers

Thus, complex numbers extend the real numbers & we write $(x, 0) = x$. The set is denoted by $\mathbb{C}$

Def. Let $i := (0, 1)$ (Imaginary unit)

- Then $iy = (0,1)(y,0) = (0, y) \longrightarrow$ (purely imaginary number)
- $i^2 = (0,1)(0,1) = (-1, 0) = -1$.
- $(a, b) = (a, 0) + ($

Thus, addition becomes

$$(x_1+iy_1)+(x_2+iy_2)=(x_1+x_2)+i(y_1+y_2)$$

& multiplication becomes

$$(x_1+iy_1)(x_2+iy_2)= x_1x_2+ix_1y_2+iy_1x_2+i^2y_1y_2$$
$$=(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1).$$

<u>Subtraction of two complex numbers:</u>

Let $z_1, z_2 \in \mathbb{C}$. Then the difference z_1-z_2 is the complex number z for which $z_1 = z+z_2$. Thus,

$$z_1-z_2 = (x_1-x_2)+i(y_1-y_2).$$

<u>Division of two complex numbers</u>

- The quotient $z = \frac{z_1}{z_2}$ $(z_2 \neq 0)$ is the complex number z for which $z_1 = zz_2$.

If $z = x+iy$, then

$$x_1+iy_1 = (x+iy)(x_2+iy_2) = (xx_2-yy_2)+i(xy_2+yx_2)$$

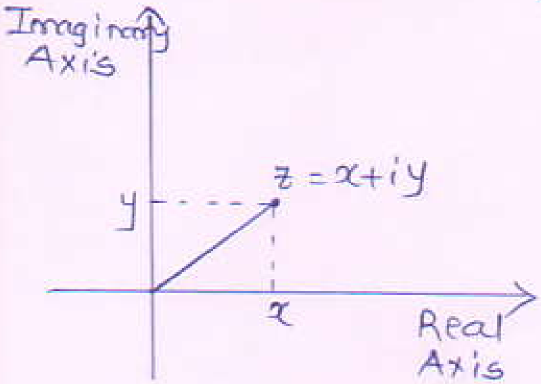
Solving for x, we get $x = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}$ and

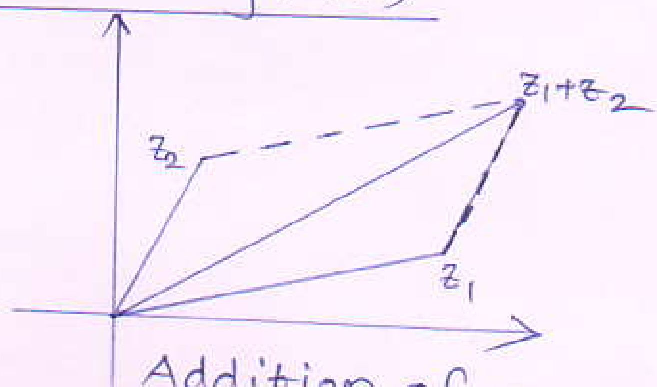
$$y = \frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}$$

- $z = \frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}$

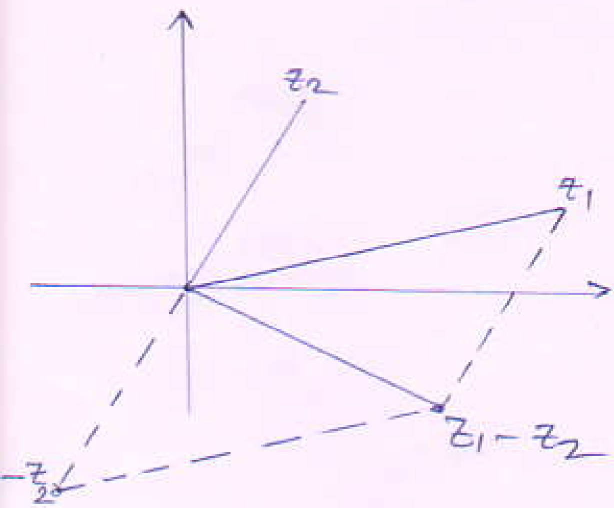
- Complex numbers satisfy the same commutative, associative and distributive laws as real numbers $\Rightarrow \mathbb{C}$ is a field.

Complex plane (Argand Diagram)





Addition of complex numbers



Complex conjugate: Let $z = x+iy$.

Then $\bar{z} = x - iy$. (Reflection of z in x-axis).

- $z\bar{z} = (x+iy)(x-iy) = x^2+y^2$ (Real !)
- $z+\bar{z} = 2x$, $z-\bar{z} = 2iy$

$\Rightarrow \mathrm{Re}(z) = \dfrac{z+\bar{z}}{2}$, $\mathrm{Im}(z) = \dfrac{z-\bar{z}}{2i}$

- $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1\, \bar{z}_2$
- $\overline{\left(\bar{z}_1\right)} = z_1$

Polar form of Complex numbers, powers and roots

- $x = r\cos\theta$, $y = r\sin\theta$.

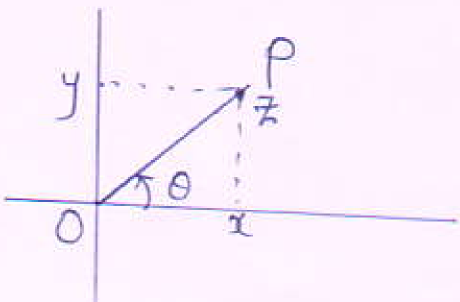
Then $z = x+iy = r(\cos\theta + i\sin\theta)$.

- $|z| = \sqrt{x^2+y^2} = \sqrt{(r\cos\theta)^2+(r\sin\theta)^2} = r$

(Absolute value) (or modulus)

- $|z| = \sqrt{z\bar{z}}$. ($|z|$: distance of z from origin)
- $|z_1 - z_2| \longrightarrow$ distance between z_1 and z_2
- $\tan\theta = \dfrac{y}{x}$ (θ: Argument of z; denoted by $\arg(z)$)

directed angle from +ve x-axis



- Angles measured in radians and +ve in counter-clockwise direction.
- For $z=0$, θ is undefined.
- For $z \neq 0$, $\arg(z)$ is unique upto integral multiples of 2π, due to the periodicity of $\sin\theta$ and $\cos\theta$.
- Principal value of θ ($\mathrm{Arg}(z)$) is such that

$$-\pi < \mathrm{Arg}\, z \leq \pi.$$

- $z = x$ (real), $x > 0 \Rightarrow \mathrm{Arg}(z) = 0$.

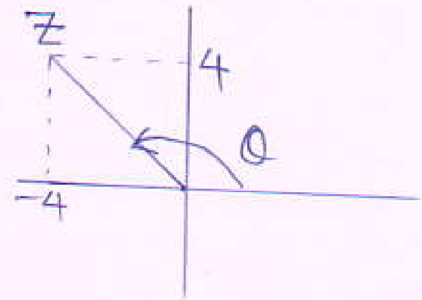
$z = x$ (real) $z < 0 \Rightarrow \mathrm{Arg}(z) = \pi$.

- $\arg(z) = \mathrm{Arg}(z) \pm 2n\pi \; (n \in \mathbb{Z})$, for $z \neq 0$

Eg. $z = -4+4i$

$|z| = \sqrt{(-4)^2+4^2} = \sqrt{32} = 4\sqrt{2}$

$\underset{\text{principal value}}{\theta} = \tan^{-1}\left(\frac{y}{x}\right) = \frac{3\pi}{4}$



$\Rightarrow$

$z = 4\sqrt{2}\left(\cos\left(\frac{3\pi}{4}+2n\pi\right)+i\sin\left(\frac{3\pi}{4}+2n\pi\right)\right).$

Warning: Let $\theta_1 = \arg(1+i)$, $\theta_2 = \arg(-1-i)$.

$\tan\theta_1 = \tan\theta_2 = 1$.

Triangle Inequality. $\boxed{\text{No ordering of complex numbers}}$.

$|z_1+z_2| \leq |z_1| + |z_2|$

Generalization: $|z_1+\dots+z_n| \leq |z_1|+\dots+|z_n|$.

Multiplication & Division in polar form

- $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ $\quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$
- $z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$ (Prove!)

$\Rightarrow |z_1 z_2| = r_1 r_2 = |z_1||z_2|$

& $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ (up to multiples of 2π)

- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$ Since $z = \frac{z_1}{z_2}\cdot z_2$ so that $|z| = \left|\frac{z_1}{z_2}\right||z_2|$
- $\arg(z_1) = \arg\left(\frac{z_1}{z_2}\cdot z_2\right) = \arg\left(\frac{z_1}{z_2}\right) + \arg(z_2)$

$\Rightarrow$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

Example $z_1 = -2+2i$, $z_2 = 3i$

$z_1 z_2 = -6-6i$ $\quad \frac{z_1}{z_2} = \frac{(-2+2i)(-3i)}{(3i)(-3i)} = \frac{2}{3} + i\frac{2}{3}$.

$|z_1 z_2| = \sqrt{(-2)^2+2^2}\sqrt{3^2} = 6\sqrt{2}$

$\left|\frac{z_1}{z_2}\right| = \frac{2}{3}\sqrt{2}$

$\text{Arg}(z_1) = \frac{3\pi}{4}$, $\text{Arg}(z_2) = \frac{\pi}{2}$, $\text{Arg}(z_1 z_2) = -\frac{3\pi}{4}$

$\text{Arg}\,\frac{z_1}{z_2} = \frac{\pi}{4}$