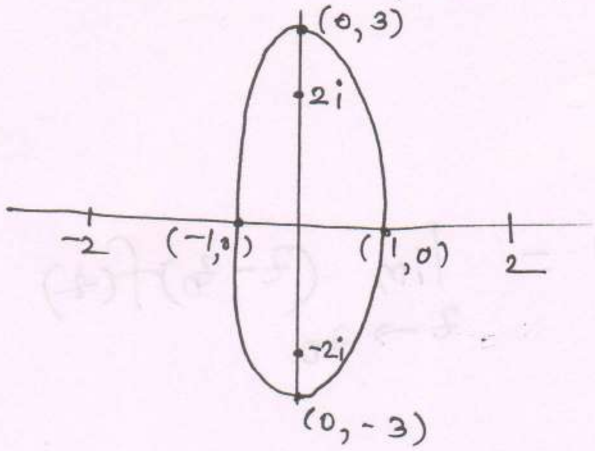
Example

Evaluate the integral $\oint_C \left(\frac{z e^{\pi z}}{z^4 - 16} + z e^{\pi/z} \right) dz$,

where C is the ellipse $9x^2 + y^2 = 9$ (counterclockwise)



$$\oint_C \left(\frac{z e^{\pi z}}{z^4 - 16} + z e^{\pi/z} \right) dz = \oint_C \frac{z e^{\pi z}}{z^4 - 16}\, dz + \oint_C z e^{\pi/z}\, dz$$

$$=: \quad ① + ②$$

By residue theorem,

$$① = 2\pi i \left(\operatorname*{Res}_{z=2i} \frac{z e^{\pi z}}{z^4 - 16} + \operatorname*{Res}_{z=-2i} \frac{z e^{\pi z}}{z^4 - 16} \right) \quad \left[\begin{array}{c} \text{Simple} \\ \text{poles} \\ \text{at } z = \pm 2i \end{array} \right]$$

$$= 2\pi i \left\{ \left. \frac{z e^{\pi z}}{4 z^3} \right|_{z=2i} + \left. \frac{z e^{\pi z}}{4 z^3} \right|_{z=-2i} \right\}$$

$$= 2\pi i \left(-\frac{1}{16} - \frac{1}{16} \right) = -\frac{\pi i}{4}.$$

②: $z e^{\pi/z}$ has an essential singularity at $z=0$.

$$z e^{\pi/z} = z \left(1 + \frac{\pi}{z} + \frac{\pi^2}{2! z^2} + \cdots \right) = z + \pi + \frac{\pi^2}{2z} + \cdots$$

$$\Rightarrow \oint_C z e^{\pi z} dz = 2\pi i \left(\frac{\pi^2}{2} \right) = \pi^3 i$$

Thus $\boxed{\text{Ans. } \pi^3 i - \frac{\pi i}{4} = \pi i \left(\pi^2 - \frac{1}{4} \right)}$

- Residue integration is useful for evaluating certain <u>REAL</u> integrals!

① <u>Integrals of rational functions of $\cos\theta$ and $\sin\theta$</u>

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

$F(\cos\theta, \sin\theta)$: real rational function of $\cos\theta$ & $\sin\theta$

<u>Method</u>: (i) Let $e^{i\theta} = z$ and set

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right),$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z-\frac{1}{z}\right).$$

ii) Then F becomes a rational function in z.

ii) $\frac{dz}{d\theta} = ie^{i\theta} = iz \Rightarrow d\theta = \frac{dz}{iz}$.

Hence $I = \oint_C f(z)\,\frac{dz}{iz}$, where C: unit circle traversed counterclockwise

<u>Example</u>: Evaluate $\int_0^{2\pi} \frac{\sin\theta}{3+\cos\theta}\, d\theta$. Denote the integral by I.

Let $z = e^{i\theta}$ $\frac{dz}{iz} = d\theta$

Hence $I = \oint_C \frac{\frac{1}{2i}\left(z-\frac{1}{z}\right)}{3+\frac{1}{2}\left(z+\frac{1}{z}\right)}\,\frac{dz}{iz}$ (C: unit circle (counter-clockwise))

$$= -\oint_C \frac{z^2-1}{z(z^2+6z+1)}\, dz$$

$$= -\oint_C \frac{(z^2-1)\, dz}{z(z+3-2\sqrt{2})(z+3+2\sqrt{2})} .$$

Out of the three poles $z=0$, $z=-3+2\sqrt{2}$, $z=-3-2\sqrt{2}$, two of them, namely 0 & $-3+2\sqrt{2}$ lie within the unit circle. Hence by residue theorem,

$$I = -2\pi i \left\{ \operatorname*{Res}_{z=0} \frac{z^2-1}{z(z^2+6z+1)} + \operatorname*{Res}_{z=-3+2\sqrt{2}} \frac{z^2-1}{z(z^2+6z+1)} \right\}$$

$$= -2\pi i \left\{ \frac{0^2-1}{0^2+6(0)+1} + \frac{(-3+2\sqrt{2})^2-1}{(-3+2\sqrt{2})(-\cancel{3}+2\sqrt{2}+\cancel{3}+2\sqrt{2})} \right\}$$

$$= -2\pi i \left\{ -1 + \frac{16-12\sqrt{2}}{16-12\sqrt{2}} \right\} = 0.$$

(2) Improper integrals of rational functions

$$I = \int_{-\infty}^{\infty} f(x)\, dx = \lim_{a\to-\infty} \int_a^0 f(x)\, dx + \lim_{b\to\infty} \int_0^b f(x)\, dx$$

If both limits exist, we couple the two independent passages to $-\infty$ & ∞, and write

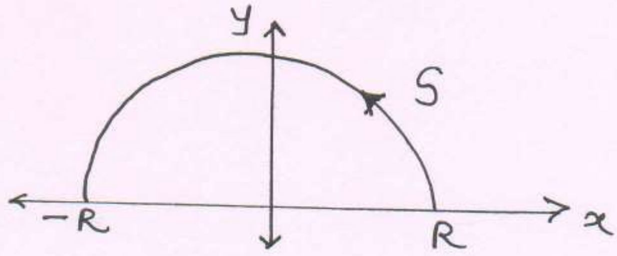
$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{R\to+\infty} \int_{-R}^{R} f(x)\, dx .$$

Conditions of f:

(1) f is a real rational function whose denominator is different from zero for all real x, that is, f does not have any poles on the real axis.

(2) $\deg(\text{Denominator}) \geq \deg(\text{Numerator}) + 2$, that is, $\deg(f(x)) \leq -2$.

Consider the corresponding contour integral $\oint_C f(z)\,dz$ around a path C shown below.



$$\oint_C f(z)\,dz = \int_S f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \text{Res} f(z),$$

where R is chosen so large that it encloses all of the poles of f in the upper-half plane (UHP).

$$\Rightarrow \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \text{Res}(f(z)) - \int_S f(z)\,dz.$$

Claim: $\lim_{R\to\infty} \int_S f(z)\,dz = 0$.

Proof: Since $\deg(f(x)) \leq -2$, for sufficiently large constants k and R_0,

$$|f(z)| < \frac{k}{|z|^2} \qquad (|z| = R > R_0)$$

$\Rightarrow$ By the ML-inequality,

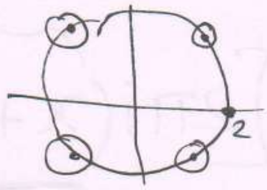
$$\left|\int_S f(z)\,dz\right| < \frac{k}{R^2}\cdot \pi R = \frac{\pi k}{R} \qquad (R > R_0)$$

$$\Rightarrow \lim_{R\to\infty} \int_S f(z)\,dz = 0.$$

Thus, $\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \text{Res} f(z)}$,

where we sum over all the residues of $f(z)$ corresponding to the poles of $f(z)$ in the UHP.

Example Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^4+16}$. Let $f(z) = \frac{1}{z^4+16}$



- $\mathrm{Deg}(f(x)) = -4 < -2$.
- No poles on the real axis.

Hence $\int_{-\infty}^{\infty} \frac{dx}{x^4+16} = 2\pi i \sum \mathrm{Res}\, f(z)$, where the sum is over all poles of f in the UHP.

Poles of f are at z where $z^4+16=0$, i.e., $z^4 = -16$, Thus they are $2e^{i\pi/4}$, $2e^{i3\pi/4}$, $2e^{-i3\pi/4}$ and $2e^{-i\pi/4}$.

Out of them, the first two lie in the UHP.

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^4+16} = 2\pi i \left\{ \mathop{\mathrm{Res}}_{z=2e^{i\pi/4}} f(z) + \mathop{\mathrm{Res}}_{z=2e^{i3\pi/4}} f(z) \right\}$$

$$= 2\pi i \left\{ \frac{1}{4z^3}\Big|_{z=2e^{i\pi/4}} + \frac{1}{4z^3}\Big|_{z=2e^{i3\pi/4}} \right\}$$

$$= 2\pi i \left\{ \frac{1}{4\times(2e^{i\pi/4})^3} + \frac{1}{4(2e^{i3\pi/4})^3} \right\}$$

$$= \frac{2\pi i}{32}\left\{ e^{-3i\pi/4} + e^{-i\pi/4} \right\} = \frac{2\pi i}{32}\left\{ e^{-\pi i + i\pi/4} + e^{-i\pi/4} \right\}$$

$$= -\frac{2\pi i}{32}\left\{ e^{i\pi/4} - e^{-i\pi/4} \right\} = -\frac{2\pi i}{32} \cdot 2i \sin\left(\frac{\pi}{4}\right)$$

$$= \frac{4\pi}{32} \cdot \frac{1}{\sqrt{2}} = \frac{\pi}{8\sqrt{2}}.$$

Consider the improper integral $\int_A^B f(x)\,dx$ s.t. $\lim\limits_{\substack{x \to \alpha \\ (A<\alpha<B)}} |f(x)| = \infty$. By defn;

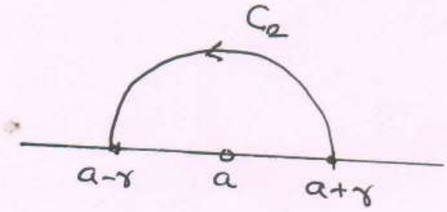
$$\int_A^B f(x)\,dx = \lim_{\varepsilon \to 0} \int_A^{\alpha-\varepsilon} f(x)\,dx + \lim_{\eta \to 0} \int_{\alpha+\eta}^{B} f(x)\,dx,$$

where both ε & η approach zero independently and through positive values.

It may be that neither limit exists if ε & $\eta \to 0$ independently, however, $\lim\limits_{\varepsilon \to 0} \left(\int_A^{\alpha-\varepsilon} f(x)\,dx + \int_{\alpha+\varepsilon}^{B} f(x)\,dx \right)$ exists. This is called the <u>Cauchy principal value</u> of the integral and is written $P.V. \int_A^B f(x)\,dx$.

Eg. $$P.V. \int_{-1}^{1} \frac{dx}{x^3} = \lim_{\varepsilon \to 0} \left[\int_{-1}^{-\varepsilon} \frac{dx}{x^3} + \int_{\varepsilon}^{1} \frac{dx}{x^3} \right] = 0.$$

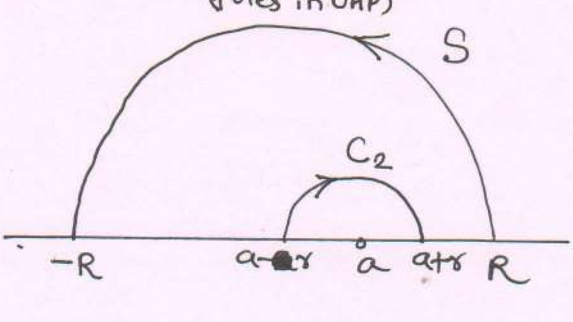
<u>Thm.</u> (i) <u>(Simple poles on the real axis)</u>



If $f(z)$ has a simple pole $z=a$ on the real axis, then

$$\lim_{\gamma \to 0} \int_{C_2} f(z)\,dz = \pi i \operatorname*{Res}_{z=a} f(z).$$

(ii) $$P.V. \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{(\text{poles in UHP})} \operatorname{Res} f(z) + \pi i \sum_{(\text{poles on real axis})} \operatorname{Res} f(z)$$



Example Find the Cauchy principal value of

$\int_{-\infty}^{\infty} \frac{x}{8-x^3}\,dx$. Let $f(x) = \frac{x}{8-x^3}$.

- $\deg(f(x)) \le -2$. ✓
- Poles at $x \ni 8 - x^3 = 0$, i.e. $(2-x)(x^2+2x+4) = 0$,

  i.e. at $x = \underset{\in \mathbb{R}}{\underset{\downarrow}{2}},\ \underset{(UHP)}{\underset{\downarrow}{-1+\sqrt{3}i}},\ -1-\sqrt{3}i$. $\Longleftarrow$ Simple poles

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x}{8-x^3}\,dx = 2\pi i \operatorname*{Res}_{z=-1+\sqrt{3}i} f(z) + \pi i \operatorname*{Res}_{z=2} f(z).$$

- $$\operatorname*{Res}_{z=-1+\sqrt{3}i} f(z) = \lim_{z \to (-1+\sqrt{3}i)} \frac{(z-(-1+\sqrt{3}i))\, z}{(2-z)(z-(-1-\sqrt{3}i))\,(z-(-1+\sqrt{3}i))}$$

$$= \frac{-1+\sqrt{3}i}{(3-\sqrt{3}i)(2\sqrt{3}i)} = \frac{-1+\sqrt{3}i}{(\sqrt{3}+3i)\cdot 2\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}} \frac{(-1+\sqrt{3}i)(\sqrt{3}-3i)}{3+9}$$

$$= \frac{1}{24\sqrt{3}}\left(-\sqrt{3}+3i+3i+3\sqrt{3}\right)$$

$$= \frac{2\sqrt{3}+6i}{24\sqrt{3}} = \frac{1+\sqrt{3}i}{12}.$$

(In a simpler way, $\operatorname*{Res}_{z=-1+\sqrt{3}i} f(z) = \left.\frac{z}{-3z^2}\right|_{z=-1+\sqrt{3}i} = \frac{1+\sqrt{3}i}{12}$

$$\operatorname*{Res}_{z=2} f(z) = \left.\frac{z}{-3z^2}\right|_{z=2} = -\frac{1}{6}.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x\,dx}{8-x^3} = 2\pi i\left(\frac{1+\sqrt{3}i}{12}\right) - \frac{\pi i}{6} = -\frac{\pi}{2\sqrt{3}}.$$

Fourier integrals

To evaluate : $\int_{-\infty}^{\infty} f(x)\cos(sx)\,dx \,, \int_{-\infty}^{\infty} f(x)\sin(sx)\,dx$ (s real)

- $\deg(f(x)) \leq -2$.

Thm. $$\int_{-\infty}^{\infty} f(x)\cos(sx)\,dx = -2\pi \sum \operatorname{Im}\left(\operatorname{Res}\left[f(z)e^{isz}\right]\right)$$

$$\int_{-\infty}^{\infty} f(x)\sin(sx)\,dx = 2\pi \sum \operatorname{Re}\left(\operatorname{Res}\left[f(z)e^{isz}\right]\right).$$

Example

Prove that $$\int_{-\infty}^{\infty} \frac{\cos(sx)}{k^2+x^2}\,dx = \frac{\pi}{k} e^{-ks}$$

$$\int_{-\infty}^{\infty} \frac{\sin(sx)\,dx}{k^2+x^2} = 0 \qquad (s>0, k>0).$$

Proof :- Consider $\frac{e^{isz}}{k^2+z^2}$. It has only one pole, namely $z = ik$, in the UHP.

$$\operatorname*{Res}_{z=ik} \frac{e^{isz}}{k^2+z^2} = \left.\frac{e^{isz}}{2z}\right|_{z=ik} = \frac{e^{-ks}}{2ik}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{isx}}{k^2+x^2}\,dx = 2\pi i \cdot \frac{e^{-ks}}{2ik} = \frac{\pi e^{-ks}}{k}.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos(sx)}{k^2+x^2}\,dx = \frac{\pi e^{-ks}}{k} \,, \quad \int_{-\infty}^{\infty} \frac{\sin(sx)\,dx}{k^2+x^2} = 0.$$