- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ —— (*)

(This is interpreted as follows: If values of two of the three multiple-valued arguments are specified, then there is a value of the third such that the equation holds.)

Eg: Suppose $\arg(z_1 z_2) = (\theta_1 + \theta_2) + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$
and $\arg(z_1) = \theta_1 + 2n_1\pi \quad (n \in \mathbb{Z})$.

Then $(\theta_1 + \theta_2) + 2n\pi = (\theta_1 + 2n_1\pi) + (\theta_2 + 2(n-n_1)\pi)$

(*) is satisfied by choosing $\arg(z_2) = \theta_2 + 2(n-n_1)\pi$.

- The statement $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$ is not always valid.

Eg. $z_1 = -1$, $z_2 = i$.

$\mathrm{Arg}(z_1) = \pi$, $\mathrm{Arg}\, z_2 = \frac{\pi}{2} \Rightarrow \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) = \frac{3\pi}{2}$.

However, $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(-i) = -\frac{\pi}{2}$.

- Find the cuberoots of $-8i$.

Ans. Note that $(-8i) = 8\left(\cos\left(-\frac{\pi}{2} + 2k\pi\right) + i\sin\left(\frac{-\pi}{2} + 2k\pi\right)\right)$
where $k \in \mathbb{Z}$.

$\Rightarrow (-8i)^{1/3} = 2\left(\cos\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right) + i\sin\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right)$

where $k = 0, 1$ & $2$.

Let the three values be $c_0$, $c_1$ & $c_2$.

$c_0 = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = \sqrt{3} - i.$

$c_1 = 2\left(\cos\left(-\frac{\pi}{6} + \frac{2\pi}{3}\right) + i\sin\left(-\frac{\pi}{6} + 2\frac{\pi}{3}\right)\right)$

$= 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 2i$

Similarly, $c_2 = -\sqrt{3} - i.$

- Let $z_1$ and $z_2$ be 2 complex numbers.

Then, $|z_1 z_2| = |z_1||z_2|$ ——— (1)

and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ ——— (2)

(up to multiples of $2\pi$)

Hence by induction, if $z = r(\cos\theta + i\sin\theta)$,

$$\boxed{z^n = r^n(\cos n\theta + i\sin n\theta)} \text{ for } n = 0,1,2,\ldots \quad \text{——— (3)}$$

- If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

then $$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

So let $z_1 = 1$ and $z_2 = z^n$. Then from (3),

$$z^{-n} = r_{\cancel{2}}^{-n}(\cos(-n\theta) + i\sin(-n\theta)).$$

This gives (3) for all integers $n$.

- Let $|z| = r = 1$ in (3). Then

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.}$$

This is known as <u>De Moivre's formula</u>.

- Prove: • $\cos 2\theta = \cos^2\theta - \sin^2\theta$
  • $\sin 2\theta = 2\sin\theta\cos\theta$

# Roots



Given $z = w^n$, $n \in \mathbb{N}$, to a given $z \neq 0$, there correspond exactly $n$ distinct values of $w$, known as $n^{th}$ roots of $z$, written as

$$w = \sqrt[n]{z} = z^{1/n}.$$

- Hence $z^{1/n}$ is a multi-valued function, in fact, $n$-valued.

- Obtaining the $n^{th}$ roots of $z$

Let $z = r(\cos\theta + i\sin\theta)$

$w = R(\cos\phi + i\sin\phi)$.

Then $w^n = z$ implies

$$R^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$

$\Rightarrow$ $R^n = r$, and hence $R = \sqrt[n]{r} > 0$ (Why?)

Also, $n\phi = \theta + 2k\pi$, $k \in \mathbb{Z}$

$$\Rightarrow \phi = \frac{\theta}{n} + \frac{2k\pi}{n}.$$

- $k = 0, 1, 2, \ldots, n-1$ give distinct values of $w$.
- Note that if $k = n$, then $\frac{2k\pi}{n} = 2\pi$.

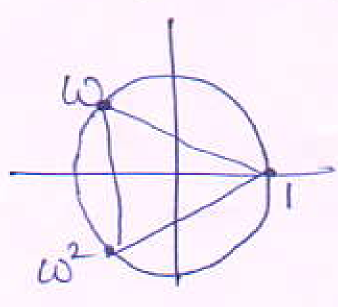
By the periodicity of sine and cosine,
$\cos\phi = \cos\left(\frac{\theta}{n} + 2\pi\right) = \cos\left(\frac{\theta}{n}\right)$ & $\sin\phi = \sin\left(\frac{\theta}{n}\right)$

Hence, $\boxed{\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)}$

- For $z=1$, $|z| = r = 1$, hence $\operatorname{Arg} z = 0$ implies
$$\sqrt[n]{1} = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \quad k = 0,1,2,\ldots,n-1.$$
($n^{th}$ Roots of unity)
- Lie on the <u>unit circle $|z|=1$</u>.
e.g. $\sqrt[3]{1} = 1, \frac{-1}{2} \pm \frac{\sqrt{3}\,i}{2}$.

- Let $\omega$ denote the $n^{th}$ root of unity corresp-onding to $k=1$. Then the $n^{th}$ roots of unity can be written in terms of $\omega$ as
$$1, \omega, \omega^2, \ldots, \omega^{n-1}.$$

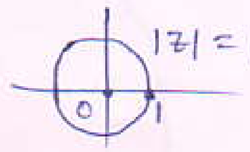
- If $z \in \mathbb{C}$ and $w_1 = \sqrt[n]{z}$, then these $n$th roots of $z$ are given by
$$w_1, w_1\omega, w_1\omega^2, \ldots, w_1\omega^{n-1}$$



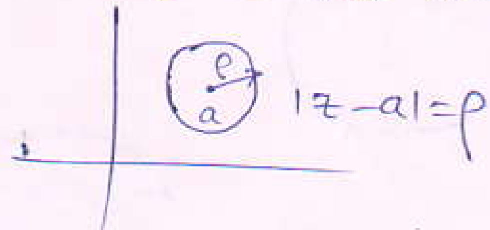

$$\omega = \frac{-1}{2} + \frac{\sqrt{3}\,i}{2}.$$

Cube-roots of unity.

- $|z| = 1$ (Unit circle) is the set of all complex numbers which are at distance 1 from the origin.




- Similarly, $|z - a| = \rho$ is the set of all $z$ whose distance $|z-a|$ from the center equals $\rho$, equivalently, the set of all $z$ whose distance from a equals $\rho$.




- Open circular disk $|z - a| < \rho \Rightarrow$

(Also known as the $\rho$-neighborhood of a)

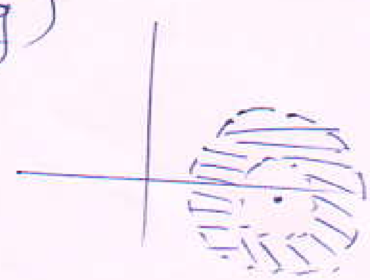
- closed circular disk

$|z - a| \leq \rho$

- exterior of a closed circular disk $|z - a| > \rho$


- open-circular disk is a neighborhood of a. ($|z-a| < \rho$, $\rho > 0$)

- Open annulus (circular ring) $\rho_1 < |z - a| < \rho_2$



- closed annulus $\rho_1 \leq |z - a| \leq \rho_2$