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MA 502 : SEM. II (PART 2) - LECTURE 2

The index of a closed curve

Note that if $\gamma(t) = a + e^{int}$ $(0 \le t \le 2\pi)$, then

$$\int_\gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{in\, e^{int}}{e^{int}}\, dt = 2\pi i n.$$

However, this evaluation is not dependent on the path γ above.

Thm. 5.1 If $\gamma : [0,1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{\gamma\}$, then $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ is an integer.

Proof: We prove this only in the case when γ is smooth. Then let $g : [0,1] \to \mathbb{C}$ be defined by

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s) - a}\, ds.$$

Obviously, $g(0) = 0$ & $g(1) = \int_0^1 \frac{\gamma'(s)\, ds}{\gamma(s) - a} = \int_\gamma \frac{dz}{z-a}$

Also, $g'(t) = \frac{\gamma'(t)}{\gamma(t) - a}$ for $0 \le t \le 1$.

But then
$$\frac{d}{dt} e^{-g(t)}(\gamma(t) - a) = e^{-g(t)}\gamma'(t) - g'(t) e^{-g(t)}(\gamma(t) - a)$$
$$= e^{-g(t)}\{\gamma'(t) - \gamma'(t)\} = 0.$$

$\Rightarrow e^{-g(t)}(\gamma(t) - a)$ is a constant function given by
$e^{-g(0)}(\gamma(0) - a) = \gamma(0) - a = e^{-g(1)}(\gamma(1) - a)$.

Since γ is closed, $\gamma(0) = \gamma(1)$.

$\Rightarrow e^{-g(1)} = 1 \Rightarrow g(1) = 2\pi i k$ for some integer k. $\square$

Defn. Let γ be a closed rectifiable curve in $\mathbb{C}$ & let $a \notin \{\gamma\}$. Then $n(\gamma; a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ is called the index of γ with respect to a or the winding number of γ around a.

BASIC PROPERTIES

If γ and σ are closed rectifiable curves having the same initial points, then

(i) $n(\gamma;a) = -n(-\gamma;a)$ for every $a \notin \{\gamma\}$.

(ii) $n(\gamma+\sigma;a) = n(\gamma;a) + n(\sigma;a)$ for every $a \notin \{\gamma\} \cup \{a\}$.

<u>Proof of (i):</u>

$$-n(-\gamma;a) = \underbrace{\frac{-1}{2\pi i}\int_{-\gamma} \frac{dz}{z-a} = -\left(\frac{-1}{2\pi i}\int_{\gamma} \frac{dz}{z-a}\right)}_{\left(\because \int_\gamma f = -\int_{-\gamma} f\right)} = n(\gamma;a)$$

REMARKS: ① If γ is smooth,

$$\int_\gamma \frac{dz}{z-a} = \int_0^1 \frac{\gamma'(t)\,dt}{\gamma(t)-a}.$$ Suppose we evaluate this to be

$$\log[\gamma(t)-a]\Big|_0^1 = \log(\gamma(1)-a) - \log(\gamma(0)-a) = 0$$
$$(\because \gamma(1) = \gamma(0)).$$

The mistake here is ignoring the fact that $\log(\gamma(t)-a)$ is the complex logarithm. If γ wraps around the point a, then we cannot define $\log(\gamma(t)-a)$ since there is no analytic branch of the logarithm defined on $\mathbb{C}-\{a\}$. (See the HW problem/Tutorial problem with $a=0$ done previously)

② One can give a correct and an intuitive (though non-rigorous) interpretation of ①. Let's use $\log z = \log|z| + i\arg z$. Then

$$\int_\gamma \frac{dz}{z-a} = \log(\gamma(1)-a) - \log(\gamma(0)-a)$$
$$= \{\log|\gamma(1)-a| - \log|\gamma(0)-a|\}$$
$$+ i\{\arg(\gamma(1)-a) - \arg(\gamma(0)-a)\}.$$

Note that

$$\log(|\gamma(1)-a|) - \log(|\gamma(0)|-a|) = 0 \quad (\because \gamma(1) = \gamma(0)).$$

The problem lies with the second part involving arg.

Since $\gamma(1)=\gamma(0)$, we see that $\arg(\gamma(1)-a)-\arg(\gamma(0)-a)$ must be an integral multiple of 2π, and moreover, this integer counts the number of times γ wraps around a; hence winding number!

Defn. A subset D of a metric space X is a component of X if it is a maximal connected subset of X, that is, D is connected and there is no connected subset of X that properly contains D.

Eg. Let $X=\{z\in\mathbb{C}: |z|\le 1\}\cup\{z: |z-3|<1\}$
Then each of the 2 sets, whose union is X, are components of X.

Some properties of components of X:

Let (X,d) be a metric space. Then

(a) Each x_0 in X is contained in a component of X.
(b) Distinct components of X are disjoint.
(c) Any component of X is closed.
(d) Let G be open in $\mathbb{C}$; then the components of G are open and there are only a countable number of them.

~~Thm 5.2~~

Let γ be a closed rectifiable curve and consider the open set $G=\mathbb{C}-\{\gamma\}$. Since $\{\gamma\}$ is compact, $\{z: |z|>R\}\subset G$ for some sufficiently large R. Thus, G has one and only one unbounded component.

Thm. Let γ be a closed rectifiable curve in $\mathbb{C}$. Then $n(\gamma;a)$ is constant for a belonging to a component of $G=\mathbb{C}-\{\gamma\}$. Also, $n(\gamma;a)=0$ for a belonging to the unbounded component of G, Proof: Define $f: G \to \mathbb{C}$ by $f(a) = n(\gamma; a)$.

Claim: f is continuous.

This, if proved, will establish the fact that f is constant on a component D of G, for, f continuous & D connected implies $f(D)$ is connected. However $f(G) \subseteq \mathbb{Z}$. Hence, $f(D)$ reduces to a single point.

To show f is continuous, note that the components of G are open. Fix $a \in G$ & let $r = d(a, \{\gamma\}) = \inf\{d(a,z) : z \in \{\gamma\}\}$
$\Rightarrow |z-a| \geq r$.

If b is a point such that $|a-b| < \delta < r/2$, then

$|z-b| = |(z-a)-(b-a)| \geq ||z-a| - |a-b||$

In particular, $|z-b| \geq |z-a| - |a-b|$

$$> r - r/2$$
$$= r/2.$$

Thus, $|f(a)-f(b)| = \frac{1}{2\pi}\left|\int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-b}\right) dz\right|$

$$= \frac{1}{2\pi}\left|\int_\gamma \frac{(a-b)}{(z-a)(z-b)} dz\right|$$

$$\leq \frac{|a-b|}{2\pi} \int_\gamma \frac{|dz|}{|z-a||z-b|}$$

$$< \frac{|a-b|}{2\pi} \cdot \frac{4}{r^2} V(\gamma)$$

$$< \frac{2\delta}{\pi r^2} V(\gamma).$$

So if $\varepsilon > 0$ is given, choosing δ to be smaller than $\frac{\pi r^2 \varepsilon}{2V(\gamma)}$ (in addition to $\delta < r/2$), we get

$|f(a)-f(b)| < \frac{2}{\pi r^2} V(\gamma) \cdot \frac{\pi r^2 \varepsilon}{2V(\gamma)} = \varepsilon$.

$\Rightarrow f$ is continuous.

Now let U be the unbounded component of G. Then $\exists R > 0 \ni U \supseteq \{z : |z| > R\}$. That is, γ lies in $B(0;R)$.
But if $|a| > R$, then $\frac{1}{z-a}$ is analytic in $B(0;R)$.
Since γ is closed, & rectifiable $n(\gamma;a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = 0$.
From part 1 of this theorem, we see that $n(\gamma;a) \equiv 0$ on U.

$\blacksquare$