Derivative

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z},$$

provided the limit exists.

- f is then said to be differentiable at z_0.
- With $\Delta z = z - z_0$, we have

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

Eg. ① $f(z) = z^3$ is differentiable $\forall\, z \in \mathbb{C}$ since

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^3 - z^3}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z^3} + 3z^2\Delta z + 3z(\Delta z)^2 + (\Delta z)^3 - \cancel{z^3}}{\Delta z}$$

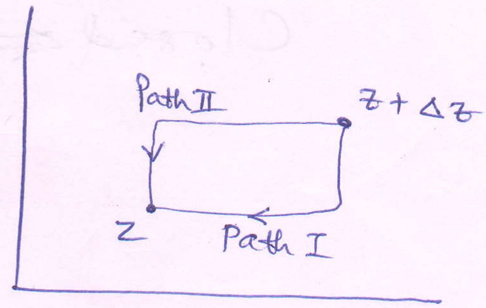
$$= 3z^2.$$

② $\bar{z}$ is not differentiable: Let $z = x+iy$ & Let $f(z) = \bar{z} = x - iy$. Suppose we write $\Delta z = \Delta x + i\Delta y$.

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0$,

$$\lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = 1$$



Analytic Functions

- A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D.

- f is said to be analytic at a point $z = z_0$ in D if f is analytic in a neighborhood of z_0.

  - Analyticity of $f(z)$ at z_0 means $f(z)$ has a derivative at every point in some neighborhood of z_0 (including z_0 itself).

  - Analytic or holomorphic mean the same

- Prove that $f(z) = |z|^2$ is nowhere analytic.

Sol^n:
$$\frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \frac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z}$$

$$= z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z} \longrightarrow (*)$$

- When $z=0$, $\displaystyle\lim_{\Delta z \to 0} \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$ exists. Hence