CAUCHY - RIEMANN EQUATIONS, LAPLACE'S EQN.

- Is there a criterion to test if the function

$$w = f(z) = u(x,y) + i\,v(x,y)$$

is analytic in some domain ?

- Yes, f is analytic in a domain D "if and only if" (iff) its real and imaginary parts satisfy the Cauchy-Riemann equations (some additional hyp. reqd.)

$$\boxed{u_x = v_y \quad \text{and} \quad u_y = -v_x}$$

$\left(u_x = \dfrac{\partial u}{\partial x},\ u_y = \dfrac{\partial u}{\partial y}, \right.$ similarly for v).

- Prove that $f(z) = z^2$ is analytic in the whole complex plane.

THM. 1 (Cauchy - Riemann equations)

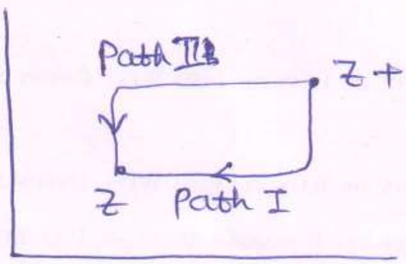
Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighborhood of $z = x+iy$ and differentiable at z itself. Then at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy - Riemann equations.

Hence $f(z) = u(x,y) + iv(x,y)$ analytic in a domain D

$$\Downarrow$$

Proof: By hypothesis, $f'(z)$ exists at z, that is,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ exists.}$$



- Let $z+\Delta z$ approach z first along path I. Let $\Delta z = \Delta x + i\Delta y$. So we first let $\Delta y \to 0$ and then $\Delta x \to 0$.

Now

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - (u(x,y) + iv(x,y))]}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$\Rightarrow \boxed{f'(z) = u_x + iv_x} \quad \text{——— (1)}$$

Similarly along path II,

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x,y+\Delta y) - u(x,y)}{i\Delta y} + i\lim_{\Delta y \to 0} \frac{v(x,y+\Delta y) - v(x,y)}{i\Delta y}$$

$$\Rightarrow \boxed{f'(z) = -iu_y + v_y} \quad \text{— (2)}$$

- Showing $\bar{z}$ is not analytic using C-R eqns.

$\bar{z} = x - iy$. So $u(x,y) = x$ and $v(x,y) = -y$

$u_x = 1$ where as $v_y = -1$. So $u_x \neq v_y$.

$\Rightarrow \bar{z}$ is not analytic. ∎

CONVERSE

THM 2: If two real-valued continuous functions $u(x,y)$ and $v(x,y)$ of two real variables x and y have continuous first ~~that~~ partial derivatives that satisfy the Cauchy-Riemann equations in some domain D, then the complex function $f(z) = u(x,y) + i\,v(x,y)$ is analytic in D.

Proof:
- Quite involved; given in Appendix in Kreyszig's book
- Continuity of partial derivatives quite crucial in the proof.

Example: Let $f(z) = z^3 + z$.

$$= (x^3 - 3xy^2 + x) + i(3x^2y - y^3 + y).$$

So
$$\left.\begin{aligned} u(x,y) &= x^3 - 3xy^2 + x \\ v(x,y) &= 3x^2y - y^3 + y \end{aligned}\right\} \text{real-valued functions}$$

$u_x = 3x^2 - 3y^2 + 1 \qquad u_y = -6xy$

$v_x = 6xy \qquad v_y = 3x^2 - 3y^2 + 1$

• If $f(z)$ is analytic in a domain D and $|f(z)| = k =$ constant in D, then $f(z) =$ constant in D.

<u>Proof</u>: $k^2 = |f(z)|^2 = u^2 + v^2$ (Note that $f = u + iv$)

$\Rightarrow uu_x + vv_x = 0$ & $uu_y + vv_y = 0$

By C-R equations,

$$uu_x - vu_y = 0 \quad \text{and} \quad uu_y + vu_x = 0$$

$\Rightarrow u^2 u_x - uvu_y = 0$ and $uvu_y + v^2 u_x = 0$

Adding the two gives,

$$u_x(u^2 + v^2) = 0 \quad \text{and similarly} \quad (u^2+v^2)u_y = 0.$$

If $k^2 = 0$, then $u^2 + v^2 = 0 \Rightarrow u = v = 0$, so f is constant

If $k \neq 0$, then $u_x = u_y = 0$ and by C-R,

$$v_x = v_y = 0 \text{ too.}$$

$\Rightarrow u =$ constant & $v =$ constant

$\Rightarrow f$ is constant.

<u>Laplace's equation</u> & <u>Harmonic functions</u>

<u>Thm. 3</u> (Laplace's eqn.)

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u and v satisfy Laplace's eqn.

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

$$\& \quad \nabla^2 v = v_{xx} + v_{yy} = 0.$$

Proof: By C-R, $u_x = v_y$

$\Rightarrow u_{xx} = v_{yx}$ —— ⓐ

(differentiate first w.r.t. y & then w.r.t. x)

Similarly $u_y = -v_x$

$\Rightarrow u_{yy} = -v_{xy}$ —— ⓑ

Assuming for now, that the derivative of an analytic function is itself analytic, we find that u and v has continuous partial derivatives of all orders, in particular, the mixed second derivatives are equal: $v_{yx} = v_{xy}$ —— ⓒ

From ⓐ, ⓑ & ⓒ, we have

$$u_{xx} + u_{yy} = 0$$

Similarly $v_{xx} + v_{yy} = 0$,

- Solutions of Laplace's eqn. having continuous second-order partial derivatives are called <u>harmonic functions</u> (very useful in <u>potential theory</u>)
- Hence the real & imaginary parts of an analytic fn. are harmonic functions.
- If two harmonic functions u and v satisfy the C-R eqns. in a domain D, they are the real & imaginary parts of an analytic function f in D