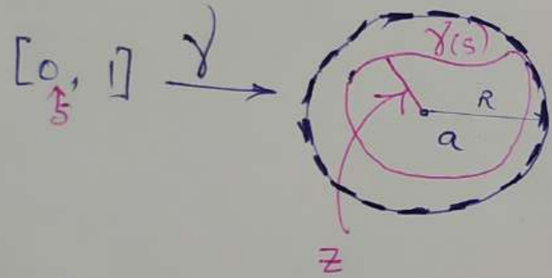
Lecture 5

① The homotopic version of Cauchy's theorem and simple connectivity

- In this section, we present a condition on a closed curve $\ni \int_\gamma f = 0$ for an analytic function f.
- While this condition is less general, it is more geometric than the condition $n(\gamma_1; w) = 0$ (or $\sum_{j=1}^{m} n(\gamma_j; w) = 0$) that we had in the previous part
- It is also used to introduce the concept of a simply connected region where Cauchy's theorem is valid
  - for every analytic fn. f
  - for every closed rectifiable curve γ.
- Let $G = B(a; R)$ and $\gamma : [0,1] \to G$ be a closed rectifiable curve.

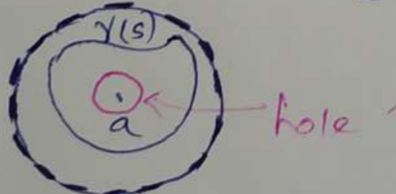


If $0 \le t \le 1$ and $0 \le s \le 1$ & $z = ta + (1-t)\gamma(s)$, then z lies on the straight line segment from a to $\gamma(s)$.
Then z lies in G.

Let $\gamma_t(s) = ta + (1-t)\gamma(s)$ for $0 \le s \le 1$ & $0 \le t \le 1$.

Then $\gamma_0 = \gamma$ and $\gamma_1 = a$; γ_t lies somewhere in between.

It was possible to come down from $\gamma(s)$ to a only because there were no holes. Imagine



<u>Defn.</u> Let $\gamma_0, \gamma_1 : [0,1] \to G$ be 2 <u>closed</u> rectifiable curves in a region G; then <u>γ_0 is homotopic to γ</u> in G if there is a continuous function $\Gamma : [0,1] \times [0,1] \to G$ such that

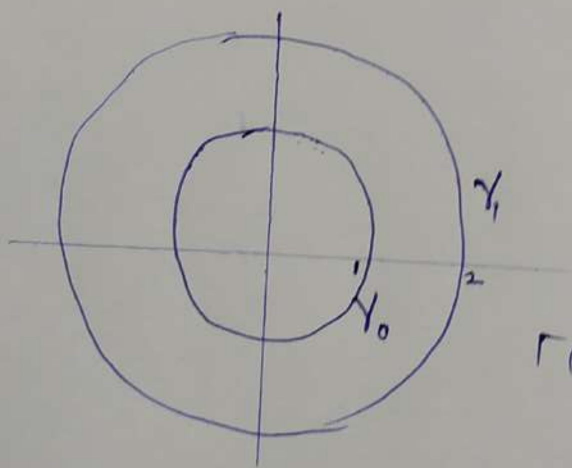
$$\left.\begin{array}{l} \Gamma(s,0) = \gamma_0(s) \quad , \Gamma(s,1) = \gamma_1(s) \qquad , (0 \le s \le 1) \\ \underbrace{\Gamma(0,t) = \Gamma(1,t),}_{\text{implies closed curves}} \quad (0 \le t \le 1). \end{array}\right] (*)$$

Now define $\gamma_t : [0,1] \to G$ by $\gamma_t(s) = \Gamma(s,t)$.
Then γ_t is a closed curve for each t. They form a continuous family of curves starting at γ_0 and going to γ_1.

- While it's not required that γ_t be rectifiable for each $t \ni 0 < t < 1$, in practice, each of them will not only be rectifiable but also smooth.

- <u>Examples</u>

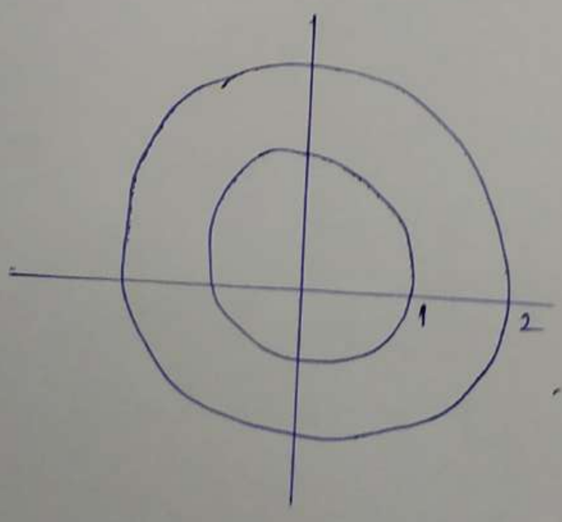
①



$G = \mathbb{C}$
$\gamma_0(s) = e^{2\pi i s}, \quad 0 \le s \le 1$
$\gamma_1(s) = 2e^{2\pi i s}, \quad 0 \le s \le 1.$

γ_0 is homotopic to γ_1
$\Gamma(s,t) = (1+t)e^{2\pi i s}, \quad 0 \le s \le 1$
$0 \le t \le 1.$

②



$G = \mathbb{C}$
$\gamma_0(s) = e^{2\pi i s}, \quad 0 \le s \le 1$
$\gamma_1(s) = -2e^{2\pi i s}, \quad 0 \le s \le 1$
γ_0 is homotopic to γ_1.
$\Gamma(s,t) = (1+t)\, e^{2\pi i (s + t/2)}.$

(3) Defn. If γ_0 is homotopic to γ_1, in G, we write $\gamma_0 \sim \gamma_1$ (ideally, we should write $\gamma_0 \sim \gamma_1 (G)$).

If the range of Γ is not required to be in G, then all curves would be homotopic.

- $\sim$ is an equivalence relation.

(a) $\gamma_0 \sim \gamma_0$ clearly.

(b) If $\gamma_0 \sim \gamma_1$ & $\Gamma : [0,1] \times [0,1] \to G$ satisfies (*), then $\gamma_1 \sim \gamma_0$ as can be seen by taking $\Lambda(s,t) = \Gamma(s, 1-t)$.

(c) If $\gamma_0 \sim \gamma_1$ & $\gamma_1 \sim \gamma_2$ with Γ satisfying (*) & $\Lambda : [0,1] \times [0,1] \to G$ satisfying (*) (with Γ replaced by Λ), then define $\Phi : [0,1] \times [0,1] \to G$ by

$$\Phi(s,t) = \begin{cases} \Gamma(s, 2t) & \text{if } 0 \le t \le 1/2 \\ \Lambda(s, 2t-1), & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Note that Φ is well-defined & continuous on $[0,1] \times [0,1]$. (Especially note that at $t = 1/2$, $\Phi(s, \frac{1}{2}) = \Gamma(s, 1) = \Lambda(s, 1) = \gamma_1(s)$),

$\Phi(s,0) = \Gamma(s,0) = \gamma_0(s)$

$\Phi(s,1) = \Lambda(s,1) = \gamma_2(s)$

$$\Phi(0,t) = \begin{cases} \Gamma(0, 2t), & \text{if } 0 \le t \le 1/2 \\ \Lambda(0, 2t-1), & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

$$\Phi(1,t) = \begin{cases} \Gamma(1, 2t), & \text{if } 0 \le t \le 1/2 \\ \Lambda(1, 2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

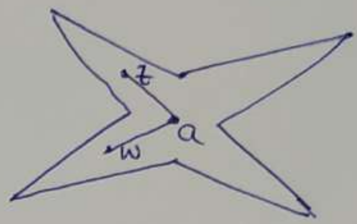
Then $\Phi(0,t) = \Phi(1,t)$.

Defn. (1) A set G is convex if for any $a, b \in G$, the line segment joining a and b, namely $[a, b]$, lies entirely in G.

(2) A set G is star-shaped if $\exists\, a \in G \ni [a, z] \in G$ $\forall z \in G$.

convex $\longrightarrow$ star-shaped

$\not\longleftarrow$ (not true)



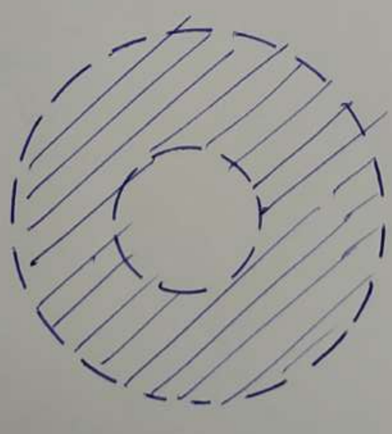
(3) G is a-star-shaped if $[a, z] \in G$ whenever $z \in G$.

(Each star-shaped set is connected (since any two points are polygonally path-connected.)

Thm. 5.9 Let G be an open set which is a-star-shaped. If γ_0 is the curve constantly equal to 'a', then every closed rectifiable curve in G is homotopic to γ_0.

Proof: Let γ_1 be a closed rectifiable curve in G. Let $\Gamma(s, t) = t\gamma_1(s) + (1-t)a$. Since G is a-star shaped, $\Gamma(s, t) \in G$ for $0 \le s, t \le 1$. Then Γ clearly satisfies (*) $\boxtimes$

Example



Not convex

Not star-shaped

But connected.

(5) <u>Defn.</u> If γ is a closed rectifiable curve in G, then <u>γ is homotopic to zero ($\gamma \sim 0$)</u> if γ is homotopic to a constant curve.

<u>Thm, 5.10</u> Cauchy's theorem (2nd version)
If $f : G \to \mathbb{C}$ is an analytic function and γ is a closed rectifiable curve in G, such that $\gamma \sim 0$, then $\int_\gamma f = 0$.

(<u>Incomplete proof</u>):
<u>Goal</u>: To show $\gamma \sim 0$ implies $n(\gamma; w) = 0 \; \forall w \in \mathbb{C} \setminus G$.

Let $\gamma_1 = \gamma$ and γ_0 be a constant curve $\ni \gamma_1 \sim \gamma_0$. Let Γ satisfy (*). Define $h(t) = n(\gamma_t; w)$, where $\gamma_t(s) = \Gamma(s,t)$ for $0 \le s, t \le 1$ & $w \in \mathbb{C} \setminus G$.
Then one can show h to be a continuous function on $[0,1]$.
Since h is integer-valued with $h(0) = 0$ ($\because \gamma_0 =$ constant $\in G$ & $n(\text{constant}; w) = 0 \; \because w \in \mathbb{C} \setminus G$), we have $h(t) \equiv 0$,
In particular $n(\gamma; w) = 0 \quad \forall w \in \mathbb{C} \setminus G$.

Problem: γ_t, $0 < t < 1$, may not be rectifiable.

<u>Thm. 5.11</u> (Cauchy's theorem (3rd version)
If γ_0 and γ_1 are two closed rectifiable curves in G & $\gamma_0 \sim \gamma_1$, then $\int_{\gamma_0} f = \int_{\gamma_1} f$ for every function f analytic in G.

<u>Proof</u>: <u>Case 1</u>: Γ has continuous second partial derivatives, i.e., $\dfrac{\partial^2 \Gamma}{\partial s \partial t} = \dfrac{\partial^2 \Gamma}{\partial t \partial s}$ throughout the square $[0,1] \times [0,1] =: I^2$.

⑥ Define $g(t) = \int_0^1 f(\Gamma(s,t)) \frac{\partial \Gamma}{\partial s}(s,t)\, ds$.

Then
$$\left.\begin{aligned} g(0) &= \int_0^1 f(\gamma_0(s))\gamma_0'(s)\, ds = \int_{\gamma_0} f \\ \& \quad g(1) &= \int_0^1 f(\gamma_1(s))\gamma_1'(s)\, ds = \int_{\gamma_1} f . \end{aligned}\right\} (\S)$$

By Leibnitz's rule, g has a continuous derivative
$$g'(t) = \int_0^1 \left(f'(\Gamma(s,t)) \frac{\partial \Gamma}{\partial s}\frac{\partial \Gamma}{\partial t} + f(s,t) \frac{\partial^2 \Gamma}{\partial t \partial s} \right) ds$$

But $\frac{\partial}{\partial s}\left[(f \circ \Gamma) \frac{\partial \Gamma}{\partial t} \right] = (f' \circ \Gamma) \frac{\partial \Gamma}{\partial s}\frac{\partial \Gamma}{\partial t} + (f \circ \Gamma) \frac{\partial^2 \Gamma}{\partial s \partial t}$.

$\implies g'(t) = f(\Gamma(1,t)) \frac{\partial \Gamma}{\partial t}(1,t) - f(\Gamma(0,t)) \frac{\partial \Gamma}{\partial t}(0,t)$

Since $\Gamma(1,t) = \Gamma(0,t)\ \forall t$, $g'(t) = 0\ \forall t$.

$\implies g$ is constant.

From $(\S)$, $\int_{\gamma_0} f = \int_{\gamma_1} f$.

$\boxtimes$