5) **Defn.** If γ is a closed rectifiable curve in G, then <u>γ is homotopic to zero ($\gamma \sim 0$)</u> if γ is homotopic to a constant curve.

Thm. 5.10 Cauchy's theorem (2^{nd} version)
If $f: G \to \mathbb{C}$ is an analytic function and γ is a closed rectifiable curve in G, such that $\gamma \sim 0$, then $\int_\gamma f = 0$.

(<u>Incomplete proof</u>):

<u>Goal</u>: To show $\gamma \sim 0$ implies $n(\gamma; w) = 0 \;\forall\, w \in \mathbb{C}\setminus G$.

Let $\gamma_1 = \gamma$ and γ_0 be a constant curve $\ni \gamma_1 \sim \gamma_0$. Let Γ satisfy (*). Define $h(t) = n(\gamma_t; w)$, where $\gamma_t(s) = \Gamma(s,t)$ for $0 \le s, t \le 1$ & $w \in \mathbb{C}\setminus G$.
Then one can show h to be a continuous function on $[0,1]$. Since h is integer-valued with $h(0) = 0$ ($\because \gamma_0 = \text{constant} \in G$ & $n(\text{contant}; w) = 0$ $\because w \in \mathbb{C}\setminus G$), we have $h(t) \equiv 0$. In particular $n(\gamma; w) = 0 \;\forall\, w \in \mathbb{C}\setminus G$.

<u>Problem</u>: γ_t, $0 < t < 1$, may not be rectifiable.

Thm. 5.11 (Cauchy's theorem (3^{rd} version)
If γ_0 and γ_1 are two closed rectifiable curves in G & $\gamma_0 \sim \gamma_1$, then $\int_{\gamma_0} f = \int_{\gamma_1} f$ for every function f analytic in G.

<u>Proof</u>: <u>Case 1</u>: Γ has continuous second partial derivatives, i.e., $\dfrac{\partial^2 \Gamma}{\partial s \partial t} = \dfrac{\partial^2 \Gamma}{\partial t \partial s}$ throughout the square $[0,1] \times [0,1] =: I^2$.

(6) Define $g(t) = \int_0^1 f(\Gamma(s,t)) \frac{\partial \Gamma}{\partial s}(s,t)\, ds$.

Then
$$\left.\begin{aligned} g(0) &= \int_0^1 f(\gamma_0(s))\gamma_0'(s)\,ds = \int_{\gamma_0} f \\ \& \quad g(1) &= \int_0^1 f(\gamma_1(s))\gamma_1'(s)\,ds = \int_{\gamma_1} f. \end{aligned}\right\} \quad (\S)$$

By Leibnitz's rule, g has a continuous derivative

$$g'(t) = \int_0^1 \left(f'(\Gamma(s,t)) \frac{\partial \Gamma}{\partial s} \frac{\partial \Gamma}{\partial t} + f(s,t) \frac{\partial^2 \Gamma}{\partial t \partial s} \right) ds$$

equal by our assumption

But $\frac{\partial}{\partial s}\left[(f \circ \Gamma) \frac{\partial \Gamma}{\partial t} \right] = (f' \circ \Gamma) \frac{\partial \Gamma}{\partial s} \frac{\partial \Gamma}{\partial t} + (f \circ \Gamma) \frac{\partial^2 \Gamma}{\partial s \partial t}$.

$$\Rightarrow g'(t) = f(\Gamma(1,t)) \frac{\partial \Gamma}{\partial t}(1,t) - f(\Gamma(0,t)) \frac{\partial \Gamma}{\partial t}(0,t)$$

Since $\Gamma(1,t) = \Gamma(0,t)\ \forall t$, $g'(t) = 0\ \forall t$.

$\Rightarrow g$ is constant.

From $(\S)$, $\int_{\gamma_0} f = \int_{\gamma_1} f$.

$\boxtimes$

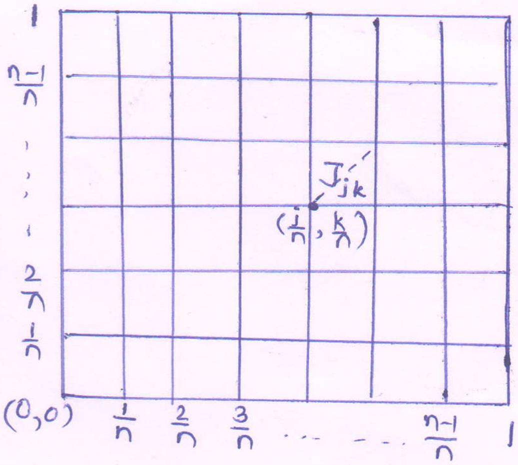
Proof of the general case Let $I = [0,1]$.

(A) Since $\gamma_0 \sim \gamma_1$, $\exists$ a continuous fn. $\Gamma : I^2 \to G \ni$ Satisfying

(*). Since I^2 is compact, Γ is uniformly continuous. Moreover, since $\Gamma(I^2)$ is a compact subset of G. So $d(\Gamma(I^2), \mathbb{C}\setminus G) > 0$. Let r denote this distance. Since Γ is uniformly continuous on I^2, $\exists\ n \in \mathbb{N} \ni$ if $\sqrt{(s-s')^2 + (t-t')^2} < 2/n$, then $|\Gamma(s,t) - \Gamma(s',t')| < r$.

— (1)

B



Divide the ~~...~~ unit square into the grid on the left.

Let J_{jk} denote the square formed by vertices $\left(\frac{j}{n}, \frac{k}{n}\right)$, $\left(\frac{j+1}{n}, \frac{k}{n}\right)$, $\left(\frac{j+1}{n}, \frac{k+1}{n}\right)$ and $\left(\frac{j}{n}, \frac{k+1}{n}\right)$ for $0 \le j, k \le n-1$

Also let Z_{jk} denote the image of $\left(\frac{j}{n}, \frac{k}{n}\right)$ under Γ for $0 \le j, k \le n$, i.e.,

$$Z_{jk} = \Gamma\left(\frac{j}{n}, \frac{k}{n}\right).$$

Since the diagonal of $J_{jk} = \sqrt{2}/n < 2/n$, we find that

$$\Gamma(J_{jk}) \subseteq B(Z_{jk}, r).$$

Let P_{jk} be the closed polygon

$$[Z_{jk}, Z_{j+1,k}, Z_{j+1,k+1}, Z_{j,k+1}, Z_{jk}];$$

Since $B(Z_{jk}, r)$ is convex, $P_{jk} \subseteq B(Z_{jk}, r)$.

Now any closed rectifiable curve in a disk, in particular, P_{jk} in $B(Z_{jk}, r)$, Cauchy's theorem is applicable.

Hence $\int_{P_{jk}} f = 0$ for any analytic function f.

$$—(2)$$

(C) Our goal now is to show $\int_{\gamma_0} f = \int_{\gamma_1} f$ by going up the ladder, one step (rung) at a time.

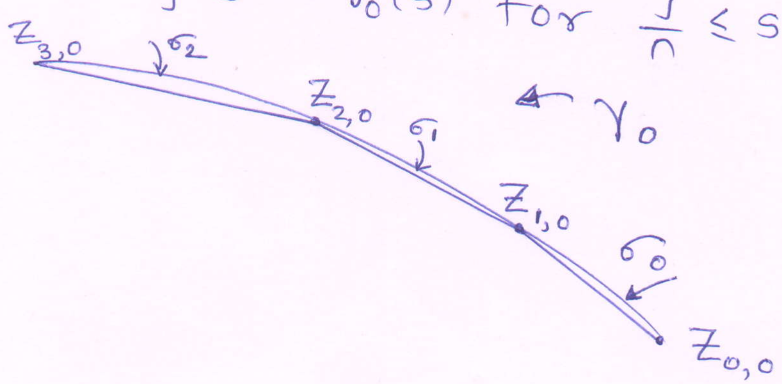
To that effect, let Q_k be the closed polygon $[z_{0,k}, z_{1,k}, \dots, z_{n,k}]$ which is a polygonal approximation of closed path $\Gamma(s, \frac{k}{n})$ for $s \in [0,1]$.

We will show that

$$\int_{\gamma_0} f = \int_{Q_0} f = \int_{Q_1} f = \dots = \int_{Q_n} f = \int_{\gamma_1} f .$$

Let us now consider the partition $\{\sigma_0(s), \sigma_1(s), \dots \sigma_{n-1}(s)\}$ of the curve $\gamma_0(s)$ where

$$\sigma_j(s) = \gamma_0(s) \text{ for } \frac{j}{n} \le s \le \frac{j+1}{n} .$$



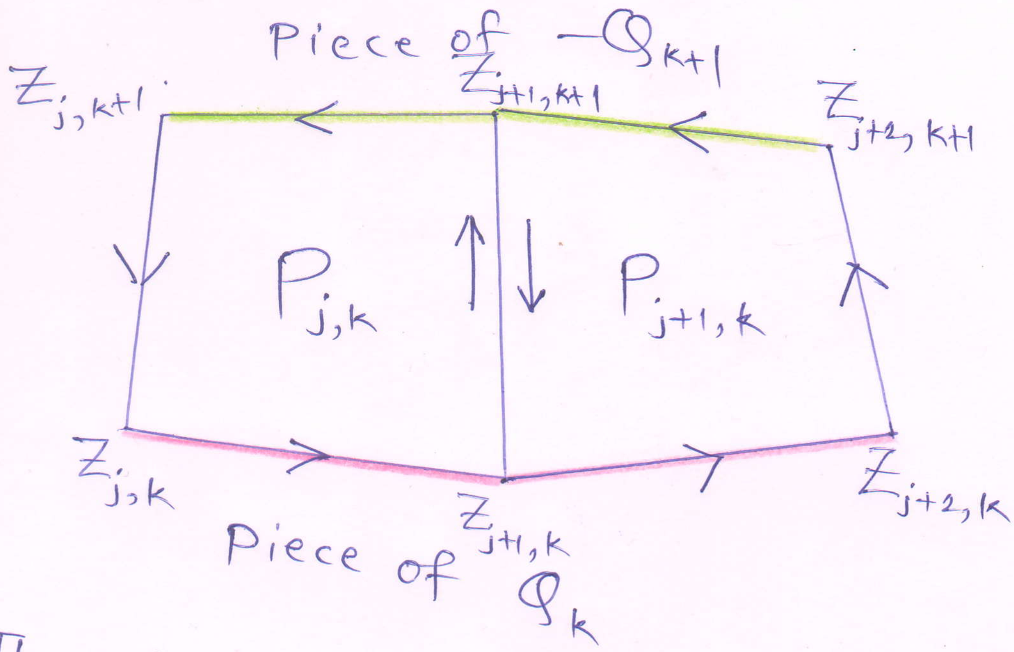
Then $\sigma_j + [z_{j+1,0}, z_{j,0}]$ is a closed rectifiable curve in the disk $B(z_{j,0}; r) \subseteq G$. Hence by Cauchy's theorem,

$$\int_{\sigma_j} f = -\int_{[z_{j+1,0}, z_{j,0}]} f = \int_{[z_{j,0}, z_{j+1,0}]} f .$$

Summing both sides over j from 0 to n, we see that $\int_{\gamma_0} f = \int_{Q_0} f$. Similarly $\int_{\gamma_1} f = \int_{Q_n} f$

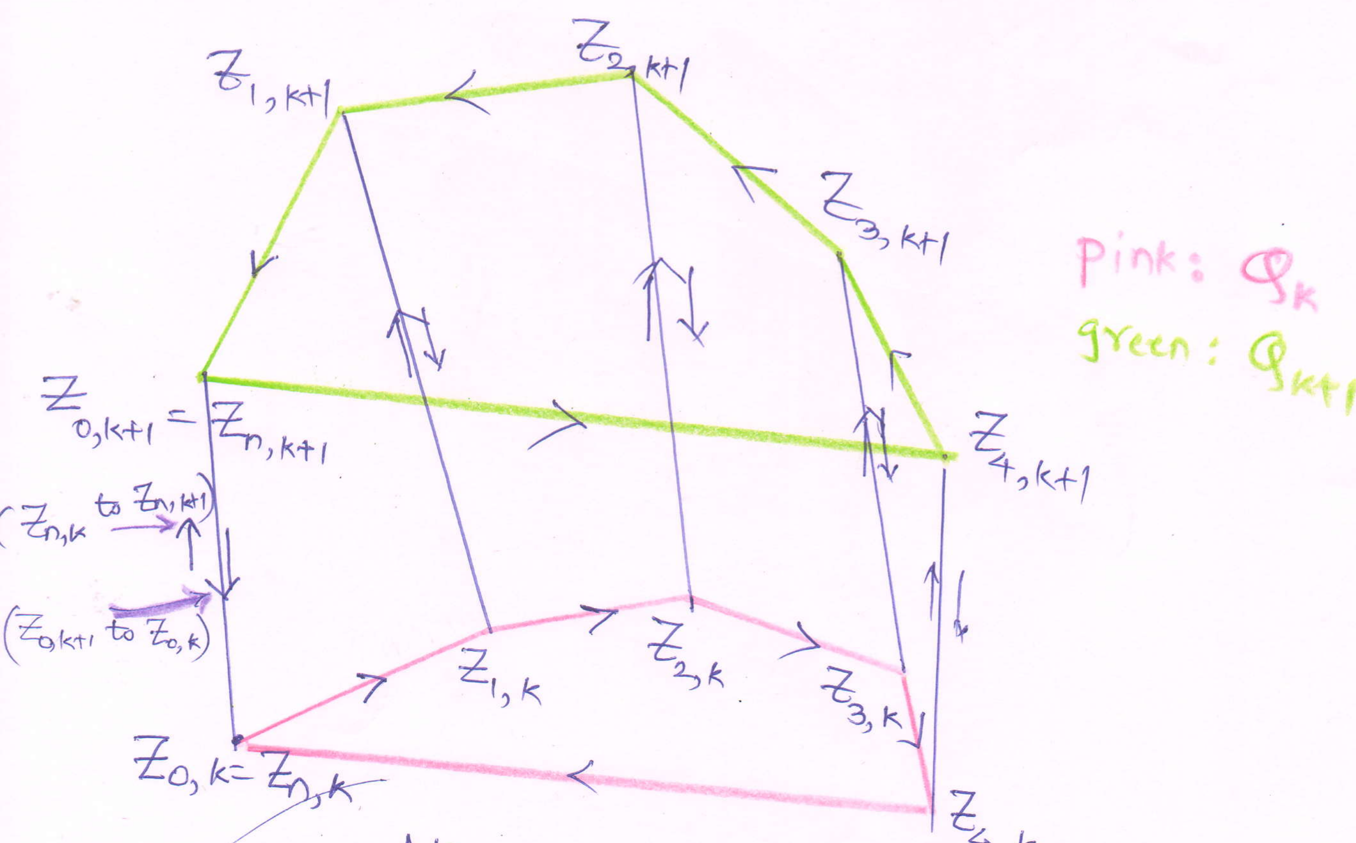
(D) Claim: $\int_{Q_k} f = \int_{Q_{k+1}} f$ for $0 \le k \le n-1$.

From (2), $\displaystyle\sum_{j=0}^{n-1} \int_{P_{jk}} f = 0$ — (3)



The right hand part of $P_{j,k}$ and the left-hand part of $P_{j+1,k}$ lead to integrals which cancel each other out.

Also, $z_{0,k} = \Gamma(0, \frac{k}{n}) = \Gamma(1, \frac{k}{n}) = z_{n,k}$ — (4)



Here $n = 5$

Note that (4) implies

$$[z_{0,k+1}, z_{0,k}] = -[z_{n,k}, z_{n,k+1}].$$

Hence $\int_{Q_k} f - \int_{Q_{k+1}} f = 0$.

$\boxtimes$

<u>Corollary</u>: Cauchy's thm. (3rd version)

$\downarrow$ implies

Cauchy's thm. (2nd version)

<u>Proof</u>: We have to show that if γ is a closed rectifiable curve in $G \ni \gamma \sim 0$, then $\int_\gamma f = 0$ for an analytic function f in G.

To see this, let $\gamma_0 = \gamma$ & $\gamma_1 :=$ a constant curve in G.

Then $\gamma_0 \sim \gamma_1$ implies $\gamma \sim 0$.

Also $\int_{\gamma_0} f = \underbrace{\int_{\gamma_1} f = 0}_{\uparrow \text{ since constant curve}}$.

$\boxtimes$