Section 13.5 – Exponential function

Defn. $e^z = e^x(\cos y + i \sin y)$

Motivation: (i) e^z should reduce to its real counterpart, when $z = x$ is real,

(ii) e^z should be an entire function, that is, analytic for all z.

(iii) Need $(e^z)' = e^z$.

(i) – Easy to prove

(ii) $u = e^x \cos y$, $v = e^x \sin y$

$u_x = e^x \cos y = v_y$ & $u_y = -e^x \sin y = -v_x$

Since this is true for all values of x & y, e^z is entire.

- $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$$e^{z_1} e^{z_2} = e^{x_1}(\cos y_1 + i\sin y_1) . e^{x_2}(\cos y_2 + i\sin y_2)$$
$$= e^{x_1+x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2))$$
$$= e^{z_1+z_2}$$

- So if $z_1 = x$ & $z_2 = iy$,

$$e^{z_1+z_2} = e^{x}. e^{iy}$$

Let $z = x+iy$. Then

$$e^{z} = e^{x+iy} = e^{x}. e^{iy}.$$

Now if $x=0$, we have Euler's formula

$$\boxed{e^{iy} = \cos y + i\sin y.} \quad \text{——} \ (*)$$

- Now the polar form of a complex number can be written in the form

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Also from (*), (a) $e^{2\pi i} = \cos(2\pi) + i\sin(2\pi)$
$= 1$

(b) $e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1$,

(c) Similarly, $e^{\pi i/2} = i$, $e^{-i\pi/2} = -i$, $e^{-\pi i} = -1$.

- Also, $|e^{iy}| = |\cos y + i\sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$.

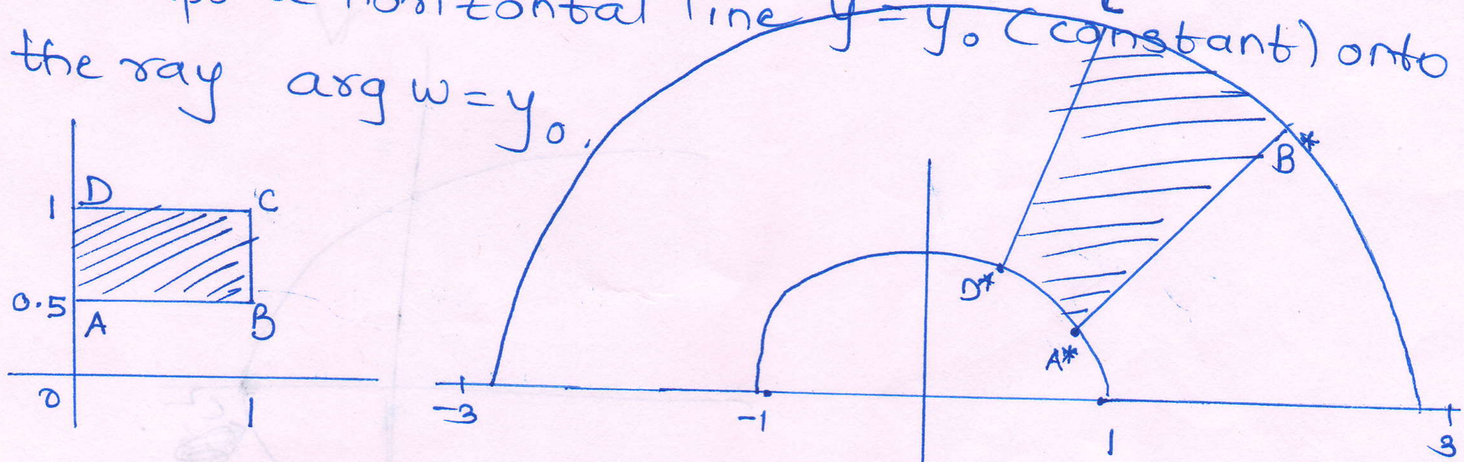
Thus, $|e^{z}| = |e^{x+iy}| = |e^{x}. e^{iy}| = e^{x}.1 = e^{x}$,

Hence $\arg(e^{z}) = y \pm 2n\pi$, $(n = 0, 1, 2, ---)$.

- $|e^z| = e^x \neq 0$. Hence $e^z \neq 0$.

Remark I: Thus e^z is an entire function that never vanishes. This is in contrast to non-constant polynomials which, though entire, always vanish at some point.

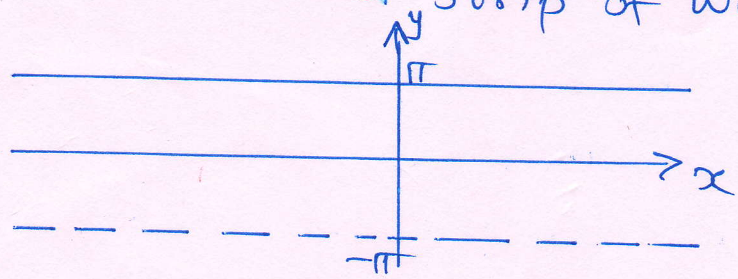
- e^z maps a vertical line $x = x_0$ (constant) onto the circle $|w| = e^{x_0}$
- e^z maps a horizontal line $y = y_0$ (constant) onto the ray $\arg w = y_0$.



<u>Periodicity of e^z with period $2\pi i$</u>

$$e^{z+2\pi i} = e^z \quad \forall z.$$

- All values assumed by $w = e^z$ are assumed if z lies in the horizontal strip of width 2π $\ni$ $-\pi < y \leq \pi$



This is called the <u>fundamental region of e^z</u>.

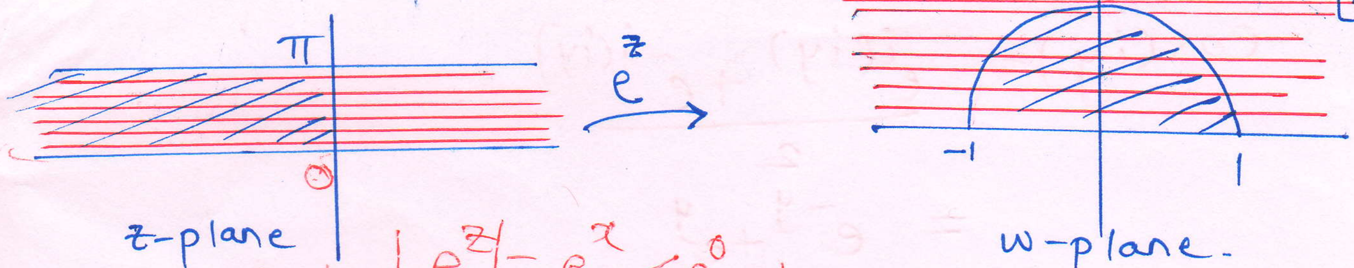
- Thus, e^z maps the fundamental region <u>bijectively</u> onto the complex plane (excluding the origin).

Mapping $w = e^z$

If $w = \rho e^{i\phi}$ & $z = x+iy$, then

$$\rho e^{i\phi} = e^{x+iy} = e^x \cdot e^{iy}$$

$$\Rightarrow \rho = e^x \text{ \& } \phi = y + 2n\pi . \ (n \in \mathbb{Z})$$



$|w| = |e^z| = e^x \leq e^0 = 1$

e^z is.

$\therefore$ $\tan z, \sec z, \cot z, \operatorname{cosec} z$ are <u>NOT</u> entire functions of z.