24/6/2020

MA 502: SEM-II (PART 2) - LECTURE 8

We have shown earlier that if an analytic function f has a zero at $z=a$, then $f(z)=(z-a)^m g(z)$, where g is analytic and $g(a)\neq 0$. Suppose G is a region and f is analytic in G with zeros at $a_1,\dots,a_m$. Then

$$\boxed{f(z)=(z-a_1)(z-a_2)\dots(z-a_m)\,g(z)}, \quad \text{—} \; (*)$$

where g is analytic on G and $g(z)\neq 0$ for any z in G.

Now

$$f'(z)=(z-a_2)\dots(z-a_m)g(z)+(z-a)\frac{d}{dz}(z-a_2)\dots(z-a_m)g(z)$$

$$\Rightarrow \boxed{\frac{f'(z)}{f(z)}=\frac{1}{z-a_1}+\frac{1}{z-a_2}+\dots+\frac{1}{z-a_m}+\frac{g'(z)}{g(z)}}, \quad \text{—} \; (**)$$

for $z\neq a_1,a_2,\dots,a_m$.

Thm. 5.18 Let G be a region and let f be an analytic function on G with zeros $a_1,\dots,a_m$ (repeated according to multiplicity). If γ is a closed rectifiable curve in G which does not pass through any point a_k and if $\gamma\approx 0$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz=\sum_{k=1}^{m} n(\gamma;a_k).$$

Proof: If $g(z)\neq 0$ for any z in G, then $\frac{g'(z)}{g(z)}$ is analytic in G. Since $\gamma\approx 0$, by Cauchy's thm. (2nd version), $\int_\gamma \frac{g'(z)}{g(z)}\,dz=0$. The result now follows from $(**)$ and the defn. of the winding number. $\boxtimes$

Cor. 5.19 Let f, G and γ be as in the preceding thm, except that $a_1, \ldots, a_m$ are points in G satisfying $f(z)=\alpha$. Then
$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)-\alpha} = \sum_{k=1}^{m} n(\gamma; a_k).$$

Examples: (1) Let γ be the circle $|z|=2$. Calculate
$$\int_\gamma \frac{(2z+1)\,dz}{z^2+z+1}.$$
Now roots of the denominator are
$$\omega_\pm = \frac{-1 \pm \sqrt{3}\,i}{2}.$$
γ does not pass through ω_+ and ω_- & encloses them. Hence by Thm. 5.18,
$$\int_\gamma \frac{2z+1}{z^2+z+1}\,dz = 2(2\pi i) = 4\pi i.$$

(2) Let $\gamma : [0,1] \to G$ be a closed rectifiable curve in $\mathbb{C}$. Suppose that f is analytic in G. Then $f\circ\gamma = \sigma$ is a closed rectifiable curve in $\mathbb{C}$ (HW problem).
If $\alpha \in \mathbb{C} \ni \alpha \notin \{\sigma\} = f(\{\gamma\})$, then
$$n(\sigma;\alpha) = \frac{1}{2\pi i}\int_\sigma \frac{dw}{w-\alpha}$$
$$= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)-\alpha}$$
$$= \sum_{k=1}^{m} n(\gamma; a_k),$$
where a_k are the points in G with $f(a_k)=\alpha$.

Thm [5.20]: Suppose f is analytic in $B(a;R)$ and let $\alpha = f(a)$. If $f(z)-\alpha$ has a zero of order m at $z=a$, then there is an $\varepsilon>0$ and $\delta>0$ s.t. for $|\zeta-\alpha|<\delta$, the equation $f(z)=\zeta$ has exactly m simple roots in $B(a;\varepsilon)$.

Remarks: ① The above theorem says $f(B(a;\varepsilon)) \supset B(\alpha;\delta)$, since if $\xi \in B(\alpha;\delta)$, then $\exists$ some roots of $f(z)=\xi$ inside $B(a;\varepsilon)$ (at least $1 \because m \geq 1$). Hence $\xi \in f(B(a;\varepsilon))$. Hence $f(B(a;\varepsilon)) \supset B(\alpha;\delta)$.

② The condition $f(z)-\alpha$ has a zero of finite multiplicity guarantees that f is not constant. — (*)

Proof: Since the zeros of an analytic function are isolated, we can choose $\varepsilon>0$ s.t. $\varepsilon<\frac{R}{2}$, $f(z)=\alpha$ has no solutions with $0<|z-a|<2\varepsilon$, and $f'(z)\neq 0$ if $0<|z-a|<2\varepsilon$

(Justification of the underlined part: $\boxed{\text{Case 1}}$: $m=1$,

Let $F(z)=f(z)-\alpha$. Then $F(z)=(z-a)g(z)$ where $g(a)\neq 0$.

$\Rightarrow F'(z)=(z-a)g'(z)+g(z)$ & $F'(z)=f'(z)$

Hence $f'(a)=g(a)\neq 0$. Since f is analytic, f' is continuous. So $\exists\, \varepsilon>0 \ni f'(z)\neq 0$ for $z\in B(a;\varepsilon)$.

$\boxed{\text{Case 2}}$ $m\geq 2$. Then $F(z)=(z-a)^m g(z)$, $g(a)\neq 0$

$\Rightarrow F'(a)=f'(a)=0$.

Now $f'\not\equiv 0$, then f' is a non-constant analytic fn; hence its zeros are isolated. In particular, a is an isolated zero of f'. $\Rightarrow \exists\, \varepsilon>0 \ni f'(z)\neq 0$ for $z\in B(a;\varepsilon)$.

If $f'\equiv 0$, then f is constant $\rightarrow\!\leftarrow$ (see (*))

Let $\gamma(t) = a + \varepsilon e^{2\pi i t}$, $0 \le t \le 1$ and put $\sigma = f\circ\gamma$.
Then $\alpha \notin \{\sigma\}$ since a is in the interior of γ, so $\alpha = f(a) \notin \{\sigma\}$. Then $\exists\ \delta > 0 \ni B(\alpha;\delta) \cap \{\sigma\} = \phi$ ($\because \{\sigma\}$ being compact is closed & so $\mathbb{C} - \{\gamma\}$ is an open set containing α).

Then $B(\alpha;\delta)$ is contained in some component of $\mathbb{C} - \{\sigma\}$, so then $|\alpha - \zeta| < \delta$ implies

$$n(\sigma;\alpha) = n(\sigma;\zeta) \quad (\text{See a thm. in Lecture } 2)$$
$$= \sum_{k=1}^{p} n(\gamma; z_k(\zeta)),$$

where z_k are the roots of $f(z) = \zeta$ in $B(a;\varepsilon)$.

Now a is a zero of order m of $f(z) - \alpha$.
$\Rightarrow n(\sigma;\alpha) = m$.

$$\Rightarrow \sum_{k=1}^{p} n(\gamma; z_k(\zeta)) = m$$

But since γ is a circle, $n(\gamma; z_k)$ must be either 0 or 1.

$\Rightarrow$ There are exactly m solutions to $f(z) = \zeta$ inside $B(a;\varepsilon)$.

Furthermore, $f'(z) \neq 0$ for $0 < |z - a| < \varepsilon$ implies that each of these roots (for $\zeta \neq \alpha$) must be simple. (Exercise).

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OPEN MAPPING THEOREM Thm. 5.21

Let G be a region and suppose that f is a non-constant analytic function on G. Then for any open set U in G, $f(U)$ is open.

Proof: If $U \subset G$ is open, then $f(U)$ is open if we can show that for each a in U there is a $\delta > 0 \ni B(\alpha;\delta) \subset f(U)$ where $\alpha = f(a)$.

This follows from the previous theorem (actually from remark 1 that is a consequence of the previous theorem) since we have an $\varepsilon > 0$ & $\delta > 0 \ni B(a;\varepsilon) \subset U$ and $f(B(a;\varepsilon)) \supset B(\alpha;\delta)$.

Defn. Let X and Ω be metric spaces. The function $f: X \to \Omega$ is said to be an open map if for any open set U in X, $f(U)$ is open in Ω.

Remark: 1 If f is a one-one and onto map, then the inverse map $f^{-1}: \Omega \to X$ exists and is defined by $f^{-1}(\omega) = x$, where $f(x) = \omega$.

2) Hence f open implies f^{-1} is continuous. In fact for $U \subset X$, $(f^{-1})^{-1}(U) = f(U)$.

Cor. 5.22 Suppose $f: G \to \mathbb{C}$ is one-one, analytic and $f(G) = \Omega$. Then $f^{-1}: \Omega \to \mathbb{C}$ is analytic and $(f^{-1})'(\omega) = \dfrac{1}{f'(z)}$, where $\omega = f(z)$.

Proof: By the open mapping theorem, f^{-1} is continuous and Ω is open. Since $z = f(f^{-1}(z))$ for each $z \in \Omega$, the result follows from a theorem proved earlier (pre-mid sem) given below:

* Let G and Ω be open subsets of $\mathbb{C}$. Suppose $f: G \to \mathbb{C}$ and $g: \Omega \to \mathbb{C}$ are continuous functions s.t. $f(G) \subset \Omega$ and $g(f(z)) = z$ for all z in G. If g is differentiable and $g'(z) \neq 0$, f is differentiable and $f'(z) = \dfrac{1}{g'(f(z))}$. If g is analytic, so is f.