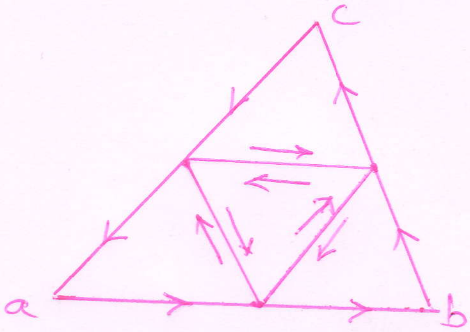
Thm 5.22 (GOURSAT'S THEOREM)

Let G be an open set and let $f: G \to \mathbb{C}$ be a differentiable function, then f is analytic on G.

Proof: It suffices to show f' is continuous on each open disk contained in G. Thus, w.l.g; we assume G itself to be an open disk.

We show $\int_T f = 0$ for every triangular path in G whence by Morera's thm; the analyticity of f is established.



Let $T = [a, b, c, a]$ and Δ = closed set formed by T and its interior.

Let Δ_j, $1 \leq j \leq 4$, be the 4 triangles T_j formed by means of each of the sides of T.

Each $T_j = \partial \Delta_j$ is a triangular path &

$$\boxed{\int_T f = \sum_{j=1}^{4} \int_{T_j} f} \quad \text{—— (1)}$$

Let $T^{(1)}$ denote the path among T_j, $1 \leq j \leq 4$ $\ni$

$$\boxed{\left|\int_{T^{(1)}} f\right| \geq \left|\int_{T_j} f\right| \quad \text{for } 1 \leq j \leq 4.} \quad \text{— (2)}$$

Let $\ell(T_j)$ denote the length of the triangular path T_j.

Then $\boxed{\ell(T_j) = \frac{1}{2}\ell(T).}$

Also $\boxed{\operatorname{diam}(T_j) = \frac{1}{2}\operatorname{diam} T.}$ — (3)

Now from (1), $\left|\int_T f\right| \leq 4\left|\int_{T'} f\right|$

Now iterate this process on $T^{(1)}$ to obtain a triangle $T^{(2)}$ with analogous properties. By induction, we get a sequence $\{T^{(n)}\}$ of closed triangular paths $\ni$ if $\Delta^{(n)} = T^{(n)} +$ its interior, then

$$\boxed{\Delta^{(1)} \supseteq \Delta^{(2)} \supseteq \Delta^{(3)} \supseteq \cdots\cdots} \quad -(4)$$

$$\left|\int_{T^{(n)}} f\right| \leq 4\left|\int_{T^{(n+1)}} f\right|$$

$$l(T^{(n+1)}) = \frac{1}{2} l(T^{(n)})$$

and $\operatorname{diam}(\Delta^{(n+1)}) = \frac{1}{2}\operatorname{diam}(\Delta^{(n)})$.

Thus, $$\boxed{\left|\int_T f\right| \leq 4^n \left|\int_{T^{(n)}} f\right|} \quad — \; *$$

$$l(T^{(n)}) = \left(\frac{1}{2}\right)^n l, \text{ where } l = l(T)$$

$$\boxed{\operatorname{diam} \Delta^{(n)} = \left(\frac{1}{2}\right)^n d, \text{ where } d = \operatorname{diam}(\Delta).} \quad -(5)$$

From (4) and (5) and Cantor's theorem, which states that a metric space (X,d) is complete iff for any sequence $\{F_n\}$ of non-empty closed sets with $F_1 \supset F_2 \supset \cdots$ and $\operatorname{diam} F_n \to 0$, $\bigcap_{n=1}^{\infty} F_n$ consists of a single point, we conclude that

$$\boxed{\bigcap_{n=1}^{\infty} \Delta^{(n)} \text{ consists of a single point } z_0.}$$

Let $\varepsilon > 0$ be given. Since f has a derivative at z_0, we know $\exists\, \delta > 0 \ni B(z_0;\delta) \subseteq G$ &

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| \leq \varepsilon \quad \text{whenever } 0<|z-z_0|<\delta.$$

Alternately, $|f(z)-f(z_0)-f'(z_0)(z-z_0)| < \varepsilon|z-z_0|$ whenever $|z-z_0|<\delta$.

Now choose $n \ni \operatorname{diam}(\Delta^{(n)}) = \left(\frac{1}{2}\right)^n d < \delta$.

Note that $z_0 \in \Delta^{(n)}$ & so if $z \in \Delta^{(n)}$,

$$|z-z_0| \le \operatorname{diam}(\Delta^{(n)}) < \delta.$$

By Cauchy's theorem,

$$\int_{T^{(n)}} f(z_0)\,dz = 0 = \int_{T^{(n)}} f'(z_0)(z-z_0)\,dz.$$

Hence

$$\left|\int_{T^{(n)}} f\right| = \left|\int_{T^{(n)}} \big(f(z)-f(z_0)-f'(z_0)(z-z_0)\big)\,dz + \int_{T^{(n)}} f(z_0)\,dz + \int_{T^{(n)}} f'(z_0)(z-z_0)\,dz\right|$$

$$\le \varepsilon \int_{T^{(n)}} |z-z_0|\,|dz|$$

$$\le \varepsilon \operatorname{diam} \Delta^{(n)}\, \ell(T^{(n)})$$

$$= \varepsilon\, d\, \ell \left(\frac{1}{4}\right)^n.$$

Along with (*), this implies

$$\left|\int_T f\right| \le 4^n \cdot \varepsilon \cdot d \cdot \ell \cdot \left(\frac{1}{4}\right)^n = \varepsilon d \ell.$$

Since ε is arbitrary, and d and ℓ are fixed,

$$\int_T f = 0.$$

$\boxtimes$

Chapter 6 – Singularities

CLASSIFICATION OF SINGULARITIES

• Isolated singularity – A function f has an isolated singularity at $z=a$ if there is an $R>0$ $\ni$ f is defined and analytic in $B(a;R)-\{a\}$ but not in $B(a;R)$

(We will use the notation $B(a;R)'$ to denoted $B(a;R)-\{a\}$.)

Notation: Let $\mathcal{A}(G)$ denote the space of all analytic functions on G.

• Examples of functions having isolated singularities

① $\frac{\sin z}{z}$, $\sin(\frac{1}{z})$, $\frac{e^z}{z-1}$, $\frac{z^2-1}{z-1}$, $\frac{1}{e^z-1}$, $\frac{z^4+1}{z^2+1}$, $\frac{z}{e^z-1}$,

• Example of a function which does not have isolated singularities:

$\log z$.

Ⓘ Removable singularity – Let f have an isolated singularity at a. The point a is called a removable singularity if $\exists$ $g: B(a;R) \to \mathbb{C}$ $\ni$ $g(z)=f(z)$ for $0<|z-a|<R$.

Eg. $\frac{\sin z}{z}$, $\frac{z^2-1}{z-1}$

Thm 6.1 If f has an isolated singularity at a, then the point $z=a$ is a removable singularity iff

$$\lim_{z\to a}(z-a)f(z)=0.$$

Proof: Suppose f is analytic in $B(a;R)'$. Define

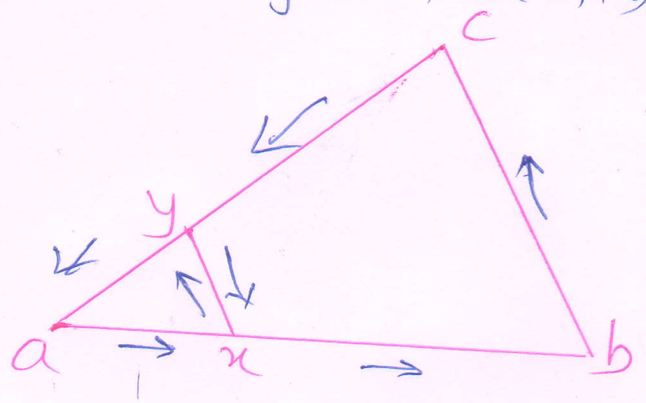
$$g(z) = \begin{cases} (z-a)f(z) & \text{for } z \neq a \\ 0 & \text{for } z = a. \end{cases}$$

Suppose $\lim_{z \to a} (z-a)f(z) = 0$. Then $\lim_{z \to a} g(z) = g(a)$,

So g is continuous on $B(a;R)$.

If we show g is analytic, it follows that 'a' is a removable singularity.

Let T be a triangle in $B(a;R)$ with $\Delta = T +$ its interior.



If $a \notin \Delta$, then $T \sim 0$ in $B(a;R)'$. So $\int_T g = 0$ by Cauchy's thm.-(2nd version, since the annulus is open and connected.)

If a is a vertex of $T = [a,b,c,a]$, and $x \in [a,b]$, $y \in [c,a]$ form $T_1 = [a,x,y,a]$ with P being the polygon $[x,b,c,y,x]$ then

$$\int_T g = \int_{T_1} g + \int_P g = \int_{T_1} g \quad (\because P \sim 0 \text{ in the punctured disk})$$

Since g is continuous and $g(a)=0$, given $\varepsilon > 0$, we can choose x & y s.t. $|g(z)| \leq \frac{\varepsilon}{\ell}$, (where ℓ is the length of T_1) for any z on T_1

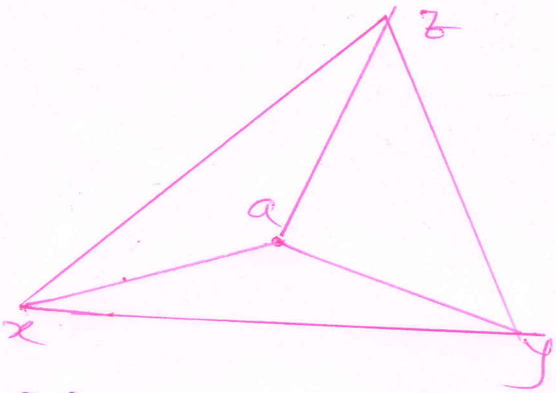
$$\Rightarrow \left|\int_T g\right| = \left|\int_{T_1} g\right| \le \frac{\varepsilon}{\ell}\cdot \ell(T_1) \le \frac{\varepsilon}{\ell}\cdot \ell = \varepsilon.$$

$$\Rightarrow \int_T g = 0.$$

If $a \in \Delta$, and $T = [x, y, z, x]$, then consider the Δ's $T_1 = [x, y, a, x]$

$T_2 = [y, z, a, y]$

$T_3 = [z, x, a, z]$,



Then $\int_{T_j} g = 0 \quad 1 \le j \le 3$.

$$\Rightarrow \int_T g = \int_{T_1} g + \int_{T_2} g + \int_{T_3} g = 0.$$

$\Rightarrow$ By Morera's thm, g is analytic.

"$\Rightarrow$" If f has a removable singularity at $z = a$, then $\exists\, h \in \mathcal{A}(B(a;R)) \ni f(z) = h(z)$ on $B(a;R)'$.

Then $\lim_{z\to a} (z-a) f(z) = \lim_{z\to a} (z-a) h(z) = 0$.

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