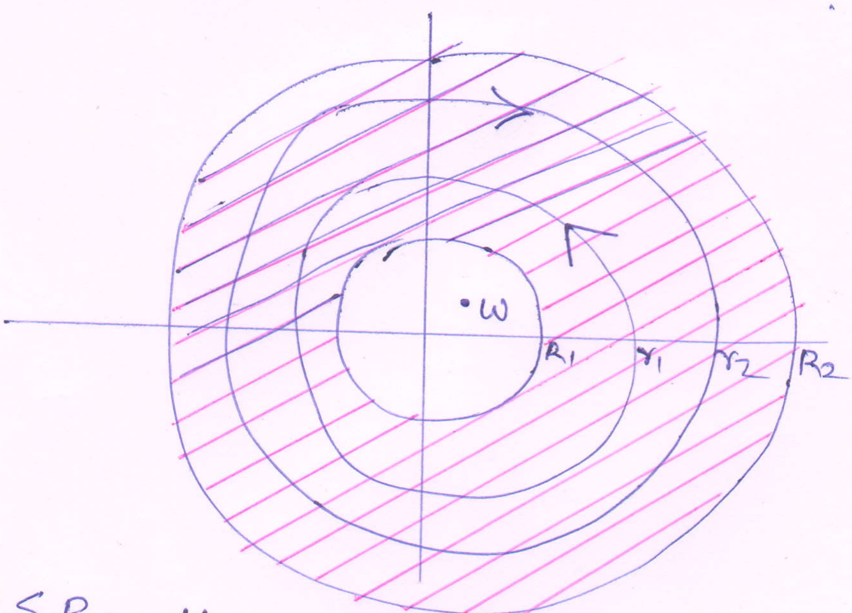
LAST TIME: Given a region $G$ and an analytic function $f$ on $G$. Is there a condition on a closed rectifiable curve $\gamma$ so that $\int_\gamma f = 0$ ?

Ans. The winding number (or the sum of winding numbers) should be zero $\forall\, w \in \mathbb{C} \setminus G$.
(This condition is also compactly written in the form $\gamma \approx 0$ "$\gamma$ is homologous to zero").

Let $G = \{z : R_1 < |z| < R_2\}$. Let $\gamma_1(t) := r_1 e^{it}$, $\gamma_2(t) = r_2 e^{-it}$ for $0 \le t \le 2\pi$, where $R_1 < r_1 < r_2 < R_2$.




If $|w| \le R_1$, then $n(\gamma_1; w) = 1$ & $n(\gamma_2; w) = -1$ — (1)
Also, if $|w| \geq R_2$, then $n(\gamma_1; w) = n(\gamma_2; w) = 0$
So $n(\gamma_1; w) + n(\gamma_2; w) = 0 \;\forall\, w \in \mathbb{C} \setminus G$.

Thm. 5.7 Let $G$ be an open subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ be an analytic function. If $\gamma_1, \gamma_2, \dots, \gamma_m$ are closed rectifiable curves in $G$ $\ni$ $n(\gamma_1; w) + \dots + n(\gamma_m; w) = 0 \;\forall\, w \in \mathbb{C} \setminus G$, then for all $a \in G \setminus \{\gamma\}$ & $k \geq 1$,

$$f^{(k)}(a) \cdot \sum_{j=1}^{m} n(\gamma_j; a) = k! \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)\, dz}{(z-a)^{k+1}}.$$

Proof: By the hypotheses,

$$f(a)\sum_{j=1}^{m} n(\gamma_j;a) = \sum_{j=1}^{m} \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(z)\,dz}{z-a}$$

Successively Differentiate both sides w.r.t. a, thereby obtaining

$$f'(a)\sum_{j=1}^{m} n(\gamma_j;a) = 1\cdot\sum_{j=1}^{m} \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(z)\,dz}{(z-a)^2}$$

$$f''(a)\sum_{j=1}^{m} n(\gamma_j;a) = (1\times 2)\sum_{j=1}^{m} \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(z)\,dz}{(z-a)^3}$$

$$\vdots$$

$$f^{(k)}(a)\sum_{j=1}^{m} n(\gamma_j;a) = k!\sum_{j=1}^{m} \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(z)\,dz}{(z-a)^{k+1}},$$

where we used Cor. 5.3 repeatedly which implies that $F_m'(a) = mF_{m+1}(a)$, where $F_m(a) = \int_\gamma \frac{f(z)\,dz}{(z-a)^m}$. for a continuous fn. f on $\{\gamma_j\}$

$\boxed{\text{Cor.}}$

When $m=1$, we have whenever $n(\gamma;w)=0$ $\forall w \in \mathbb{C}\setminus G$,

$$f^{(k)}(a)\, n(\gamma;a) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)\,dz}{(z-a)^{k+1}}.$$

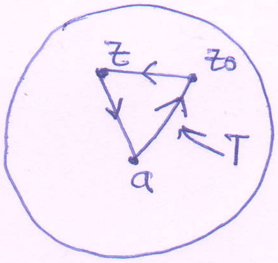
Question: Are there functions other than analytic functions that satisfy $\int_\gamma f = 0$ for all closed curves $\gamma$?

Ans. No.

Thm. 5.8 Let G be a region and let $f: G \to \mathbb{C}$ be a continuous function s.t. $\int_T f = 0$ for every triangular path T in G; then f is analytic in G.

Proof: W.l.g., we assume $G = B(a;R)$.

Goal: f has a primitive




Let $z \in G$ & define $F(z) = \int_{[a,z]} f \, dw$.

Fix $z_0 \in G$. For any point $z \in G$, by the hypothesis,

$\int_T f \, dw = 0$, where $T$ is the path shown in the above figure. Thus,

$$F(z) = \int_{[a,z_0]} f + \int_{[z_0,z]} f \text{. Hence}$$

$$\frac{F(z)-F(z_0)}{z-z_0} = \frac{1}{z-z_0}\int_{[z_0,z]} f \text{, and so}$$

$$\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{(z-z_0)}\int_{[z_0,z]} f - f(z_0)$$

$$= \frac{1}{z-z_0}\int_{[z_0,z]} (f(w)-f(z_0))\,dw, \quad \left(\text{since } \int_{[z_0,z]} f(z_0)\,dw = f(z_0)(z-z_0)\right) \quad \text{— } (*)$$

Taking absolute values on both sides of $(*)$, we have

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| \leq \sup_{w \in [z,z_0]} |f(w) - f(z_0)|$$

Now let $z \to z_0$ so that

$$\lim_{z \to z_0} \frac{F(z)-F(z_0)}{z-z_0} = f(z_0).$$

In other words, $f$ has a primitive in $G$, $\Rightarrow f$ analytic in $G$

(Since $F$ is differentiable & $f$ is continuous on $G$, $F$ is analytic $\Rightarrow F$ is infinitely differentiable (and so is $f \Rightarrow f$ is analytic on $G$)

NV

<u>Remarks:</u> (1) The triangular path isn't special for Morera's theorem to be true. The result holds for every closed rectifiable curve in $G$.

(2) In a sense, Morera's theorem is a converse of Cauchy's theorem. Combining the two, we have the following result:

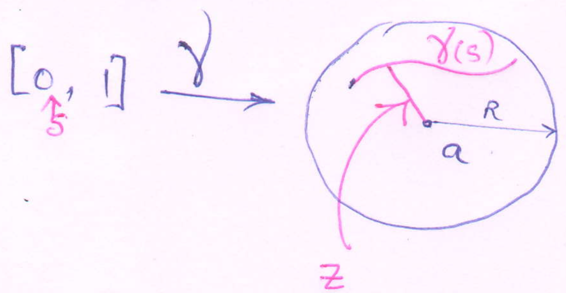
If $f$ is a continuous function in a region $G$, then $f$ is analytic in $G$ iff $\int_\gamma f = 0$ for every closed rectifiable curve in $G$.

(3) Morera's theorem is useful where one cannot directly establish analyticity of a function by the definition, for example, in the proof of Schwarz ~~(4)~~ reflection principle.

(4) The fact that the uniform limit of analytic functions is again analytic is an immediate consequence of Morera's theorem.

# The homotopic version of Cauchy's theorem and simple connectivity

- In this section, we present a condition on a closed curve $\ni \int_\gamma f = 0$ for an analytic function $f$.
- While this condition is less general, it is more geometric than the condition $n(\gamma_1; w) = 0$ (or $\sum_{j=1}^{m} n(\gamma_j; w) = 0$) that we had in the previous part
- It is also used to introduce the concept of a simply connected region where Cauchy's theorem is valid
  - for every analytic fn. $f$
  - for every closed rectifiable curve $\gamma$.
- Let $G = B(a; R)$ and $\gamma : [0,1] \to G$ be a closed rectifiable curve.




If $0 \le t \le 1$ and $0 \le s \le 1$ & $z = ta + (1-t)\gamma(s)$, then $z$ lies on the straight line segment from $a$ to $\gamma(s)$.

Then $z$ lies in $G$.

Let $\gamma_t(s) = ta + (1-t)\gamma(s)$ for $0 \le s \le 1$ & $0 \le t \le 1$.

Then $\gamma_0 = \gamma$ and $\gamma_1 = a$; $\gamma_t$ lies somewhere in between.

It was possible to come down from $\gamma(s)$ to $a$ only because there were no holes. Imagine