MA 201 - Tutorial set 1

1. Find and sketch the images of the angular region $0 \leq arg z \leq \frac{\pi}{8}$ in the case of each of the following mappings:

$$(i) w = \frac{1}{z} \;\; (ii) \;\; w = \frac{i}{z} \;\; (iii) \;\; w = z^{-2} \;\; (iv) = iz^{-2}.$$

2. Find and sketch the images of the region $y > -1$ under the following mappings:

$$(i) \;\; w = iz \;\; (ii) \;\; w = (1+i)z \;\; (iii) \;\; w = (1-i)z + 2i \;\; (iv) \;\; w = iz^2.$$

3. Let $\langle .,. \rangle$ denote the usual inner product in $\mathbb{R}^2$. In other words, if $z = (x_1, y_1)$ and $w = (x_2, y_2)$, then

$$\langle z, w \rangle = x_1 x_2 + y_1 y_2.$$

Similarly, we may define a Hermitian inner product $(.,.)$ in $\mathbb{C}$ by

$$(z, w) = z\overline{w}.$$

The term Hermitian is used to describe the fact that $(.,.)$ is not symmetric, but rather satisfies the relation

$$(z, w) = \overline{(w, z)} \text{ for all} z, w \in \mathbb{C}.$$

Show that

$$\langle z, w \rangle = \frac{1}{2} \left[(z, w) + (w, z) \right] = \text{Re}(z, w),$$

where we use the usual identification $z = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$.

4. Describe geometrically the sets of points z in the complex plane defined by the following relations:
(a) $1/z = \overline{z}$.
(b) $\text{Re}(z) = 3$.
(c) $\text{Re}(z) > c$, where $c \in \mathbb{R}$.

5. Suppose that f is analytic in a domain D. Prove that in any one of the following cases:
(a) $\text{Re}(f)$ is constant;
(b) $\text{Im}(f)$ is constant;
(c) $|f|$ is constant;
one can conclude that f is constant.

6. Fix $n \geq 1$. Show that the n^{th} roots of unity $\omega_0, \omega_1, ...\omega_{n-1}$ satisfy

a) $(z - \omega_0)......(z - \omega_{n-1}) = z^n - 1$.

b) $\omega_0 + \omega_1 + ... + \omega_{n-1} = 0$.

c)

$$\sum_{j=0}^{n-1} \omega_j^k = \begin{cases} 0 & 1 \leq k \leq n-1 \\ n & k = n. \end{cases}$$

7. Prove that

$$1 + \cos\theta + \cos 2\theta + \cos 3\theta + ... + \cos n\theta = \frac{\sin\left(\frac{1}{2}(n+1)\theta\right).\cos(\frac{n\theta}{2})}{\sin(\frac{\theta}{2})}.$$

8. Show that

$$|z_1 + z_2| \leq |z_1| + |z_2|, \quad \text{and}$$
$$||z_1| - |z_2|| \leq |z_1 - z_2| \quad \text{for all } z_1, z_2 \in \mathbb{C}.$$

9. Show that if $z \in \mathbb{C}$ lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \leq \frac{1}{3}.$$

10. Show that the only entire function $f : \mathbb{C} \to \mathbb{C}$ of the form $f(x + iy) = u(x) + iv(y)$ is given by $f(z) = \lambda z + a$ for some $\lambda \in \mathbb{R}$ and $a \in \mathbb{C}$.

11. Show that if a limit of a function $f(z)$ exists at z_0, then it must be unique.

12. Prove that the existence of the derivative of a function at a point implies the continuity of the function at that point.