MA 502 - Tutorial 11 Solutions

(by contradiction)

This prob. combines the solutions to prob. 4 of Tut. 10 & prob. 1 of Tut. 11

1. Suppose $p(z)$ is a polynomial of degree $n>0$ and $p(z)$ does not vanish for any $z \in \mathbb{C}$, then obviously, the given result

$$\frac{1}{2\pi i}\int_\gamma \frac{p'(z)}{p(z)}\,dz = \sum_{k=1}^{m} n(\gamma; a_k)$$

implies that its right-hand side is zero.

$$\boxed{\text{Claim: } \frac{1}{2\pi i}\int_\gamma \frac{p'(z)}{p(z)}\,dz = n}$$

This will give us contradiction as $n>0$. Thus FTA will be proved.

Proof of the claim: Since $p(z)$ does not vanish in $\{z : |z|>R\}$ for any $R \geq 0$, $\gamma \sim \gamma_R$, where $\gamma_R(t) = Re^{it}$, $0 \leq t \leq 2\pi$ for any $R \geq 0$.

$$\text{Hence } \frac{1}{2\pi i}\int_\gamma \frac{p'(z)}{p(z)}\,dz = \lim_{R\to\infty}\int_{\gamma_R} \frac{p'(z)}{p(z)}\,dz \quad \text{---(1)}$$

Now if p is a poly. of deg. n, then as $|z| \to \infty$,
$(1-\varepsilon)|a_n||z|^n \leq |p(z)| \leq (1+\varepsilon)|a_n||z|^n$.

In particular, with $\varepsilon = 1/2$,

$$\frac{1}{2}|a_n||z|^n \leq |p(z)| \leq \frac{3}{2}|a_n||z|^n. \quad \text{--- } (*)$$

$$\text{Consider } \left|\frac{p'(z)}{p(z)} - \frac{n}{z}\right| = \left|\frac{zp'(z) - np(z)}{zp(z)}\right|$$

If $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, then

$p'(z) = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \cdots + a_1$

$\Rightarrow \; z p'(z) - n p(z)$

$= -a_{n-1} z^{n-1} - 2a_{n-2} z^{n-2} - \cdots - (n-1) a_1 z - n a_0$,

that is, $z p'(z) - n p(z)$ is a poly. of deg. $(n-1)$.

Hence from $(*)$,

$$|z p'(z) - n p(z)| \le \frac{3}{2} |a_{n-1}| \, |z|^{n-1} \quad \text{—— } (2)$$

Also, from $(*)$,

$$|z p(z)| \ge \frac{1}{2} |a_n| \, |z|^{n+1} \quad \text{—— } (3)$$

From (2) and (3), as $|z| \to \infty$

$$\left| \frac{p'(z)}{p(z)} - \frac{n}{z} \right| \le \frac{3}{|z|^2} \frac{|a_{n-1}|}{|a_n|} \quad \text{—— } (4)$$

Now $\cancel{\int_{\gamma_R} \frac{p'(z)}{p(z)} dz = \int_{\gamma_R}}$

$$\lim_{R\to\infty} \left| \int_{\gamma_R} \frac{p'(z)}{p(z)} dz - \int_{\gamma_R} \frac{n}{z} dz \right| = \lim_{R\to\infty} \left| \int_{\gamma_R} \left(\frac{p'(z)}{p(z)} - \frac{n}{z} \right) dz \right|$$

$$\le \lim_{R\to\infty} \int_{\gamma_R} \left| \frac{p'(z)}{p(z)} - \frac{n}{z} \right| |dz|$$

$$\le \lim_{R\to\infty} \frac{3}{R^2} \frac{|a_{n-1}|}{|a_n|} \cdot 2\pi R = 0.$$

$$\Rightarrow \lim_{R\to\infty} \int_{\gamma_R} \frac{p'(z)}{p(z)} dz = \lim_{R\to\infty} \int_{\gamma_R} \frac{n}{z} dz = 2\pi i n. \quad \text{—— } (5)$$

From (1) and (5),

$$\frac{1}{2\pi i}\int_{\gamma} \frac{p'(z)}{p(z)}\,dz = n.$$

This proves the claim.

$\boxtimes$

5.1] a) $f(z) = \frac{\sin z}{z}$. It has singularity at $z=0$

Since $\lim_{z\to 0} \frac{z \cdot \sin z}{z} = \lim_{z\to 0} \sin z = \sin(0) = 0$ ~~exists~~,

f has a removable singularity at $z=0$.

Now define $f(0) = 1$

Then $f(z) = \begin{cases} \sin z / z & z \neq 0 \\ 1 & z = 0 \end{cases}$

Then $\lim_{z\to 0} \frac{\sin z}{z} = 1 = f(0)$,

Hence f is continuous at $z=0$.

Also consider $\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}$

$$= \lim_{z\to 0} \frac{(\sin z)/z - 1}{z}$$

$$= \lim_{z\to 0} \frac{\left(\frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots}{z} - 1\right)}{z}$$

$$= \lim_{z\to 0} \frac{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots - 1\right)}{z}$$

$$= 0$$

Thus $\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}$ exists

Thus f is differentiable at $z=0$.

~~Also~~ $\cancel{f'(z) = \frac{z\cos z - \sin z}{z^2}}$

~~Now~~ $\cancel{\lim_{z\to 0} f'(z)} =$

Now thus f is continuous and differentiable at $z=0$

and for $z \neq 0$, $\frac{\sin z}{z}$ is ~~a~~ an analytic function.

So $f(0)=1$ so that f is analytic at $z=0$.

(b) $f(z) = \frac{\cos z}{z}$

f has pole of order 1 at $z=0$.

Now $f(z) = \frac{\cos z}{z} = \frac{\cancel{A}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} \cdots\cdots\right)}{z}$

$= \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \cdots$

Hence singular part of f is $\frac{1}{z}$.

(3) $f(z) = \dfrac{\cos z - 1}{z}$

f has removable singularity at $z=0$
because $\lim\limits_{z\to 0} z f(z) = \lim\limits_{z\to 0} \cos z - 1 = \cos(0) - 1$

$= 1 - 1 = 0$.

f being ratio of 2 analytic fns is analytic when $z \neq 0$
To have analyticity of f at $z=0$, we define
$f(0) = 0$

Now $\lim\limits_{z\to 0} f(z) = \lim\limits_{z\to 0} \dfrac{\cos z - 1}{z} = \lim\limits_{z\to 0} \dfrac{-\sin z}{1}$

$= 0 = f(0)$

Hence f is continuous at $z=0$.

Also $\lim\limits_{z\to 0} \dfrac{f(z) - f(0)}{z - 0} = \lim\limits_{z\to 0} \dfrac{\left(\dfrac{\cos z - 1}{z} - 0\right)}{z}$

$= \lim\limits_{z\to 0} \dfrac{\cos z - 1}{z^2} = \lim\limits_{z\to 0} \dfrac{\left(1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \cdots\right) - 1}{z^2}$

$= \dfrac{-1}{2}$ exists

Hence f is differentiable at $z=0$
Hence for $z \neq 0$, as well as $z=0$, f is analytic when we define $f(0) = 0$.

(g) $f(z) = \dfrac{\log(1+z)}{z^2}$

$$= \frac{\left(z - \frac{z^2}{2} + \frac{z^3}{3} - \dots\right)}{z^2} \quad (\text{when } |z| < 1)$$

$$= \frac{\left(1 - \frac{z}{2} + \frac{z^2}{3} - \dots\right)}{z}$$

Hence f has a pole of order 1 at $z = 0$.

Now $f(z) = \dfrac{1}{z} - \dfrac{1}{2} + \dfrac{z}{3} - \dots$

Hence singular part $= \dfrac{1}{z}$

(h) $f(z) = \dfrac{1}{1 - e^z}$

Now f has singularities where $e^z = 1$

i.e. at $z = 0, \pm 2\pi i, \pm 4\pi i, \dots$

Thus f has poles at $z = 0, \pm 2\pi i, \pm 4\pi i, \dots$

~~Also $z = 0$ is pole of order 1 because~~

$\cancel{\dfrac{z}{1-e^z}}$

Also $\dfrac{1}{1 - e^z} = \dfrac{-1}{e^z - 1} = \left(\dfrac{-1}{z}\right)\left(\dfrac{z}{e^z - 1}\right)$

$$= \left(\frac{-1}{z}\right) \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \qquad (|z| < 2\pi)$$ (where B_n are Bernoulli numbers)

$$= \frac{-1}{z}\left(1 - \frac{1}{2}z + \frac{1}{12}z^2 - \cdots\right)$$

$$= \frac{-1}{z} + \frac{1}{2} - \frac{1}{12}z + \cdots$$

Hence singular part $= \frac{-1}{z}$

(j) $f(z) = z^n \sin(1/z)$.

It has essential singularity at $z=0$ because:

$$\lim_{z\to 0} \left| z^n \sin\left(\frac{1}{z}\right)\right| \neq \infty$$ (So $z=0$ is not a pole)

Also $$\lim_{z\to 0} z \cdot z^n \sin\left(\frac{1}{z}\right) \neq 0.$$ (Hence not a removable singularity)

3] $f(z) = \dfrac{8+z^2}{z^3-z^2-2z} = \dfrac{z^2+8}{z(z^2-z-2)}$

$= \dfrac{z^2+8}{z(z+1)(z-2)} = \dfrac{A}{z} + \dfrac{B}{(z+1)} + \dfrac{C}{(z-2)}$

$A = zf(z)\Big|_{z=0} = \dfrac{z^2+8}{(z+1)(z-2)}\Big|_{z=0} = \dfrac{8}{1(-2)} = -4$

$B = (z+1)f(z)\Big|_{z=-1} = \dfrac{z^2+8}{z(z-2)}\Big|_{z=-1} = \dfrac{9}{3} = 3$

$C = (z-2)f(z)\Big|_{z=+2} = \dfrac{z^2+8}{z(z+1)}\Big|_{z=2} = \dfrac{12}{2(3)} = 2$

$\therefore f(z) = \dfrac{-4}{z} + \dfrac{3}{z+1} + \dfrac{2}{z-2}$ ——— I

(a) ann (0; 1, 2)

In this annulus, we have $1 < |z| < 2$.

From I $f(z) = \dfrac{-4}{z} + \dfrac{3}{z+1} + \dfrac{2}{z-2}$

(i) Laurent's series expansion for $\dfrac{-4}{z}$ is $\dfrac{-4}{z}$ itself

(ii) Laurent's series expansion for $\frac{3}{z+1}$ is

$$\frac{3}{z+1} = \frac{3}{z(1+\frac{1}{z})} \quad \left(\text{Now } |z|>1 \therefore \left|\frac{1}{z}\right|<1\right)$$

$$= \frac{3}{z}\left[1+\frac{1}{z}\right]^{-1} = \frac{3}{z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\cdots\right]$$

(iii) Laurent's series expansion for $\frac{2}{z-2}$ is

$$\frac{2}{z-2} = \frac{-2}{2(1-\frac{z}{2})} \quad \left(\because |z|<2, \left|\frac{z}{2}\right|<\frac{1}{2}<1\right)$$

$$= \cancel{2} - \left(1-\frac{z}{2}\right)^{-1}$$

$$= -\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right)$$

6

Thus Laurent's series expansion for $f(z)$ is

$$\frac{-4}{z}+\frac{3}{z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\cdots\right]-\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

$$= \frac{-1}{z}-\frac{3}{z^2}+\frac{3}{z^3}-\frac{3}{z^4}+\cdots - 1 - \frac{z}{2}-\frac{z^2}{4}-\cdots$$

$$= \cdots - \frac{3}{z^4}+\frac{3}{z^3}-\frac{3}{z^2}-\frac{1}{z}-1-\frac{z}{2}-\frac{z^2}{4}-\frac{z^3}{8}-\cdots$$

(b) ann $(1; 0, 1)$

In this annulus, we have $0 < |z-1| < 1$.

$$f(z) = \frac{-4}{z} + \frac{3}{z+1} + \frac{2}{z-2} \quad (\text{from } \mathrm{I})$$

(i) Laurent's series expansion for $\frac{-4}{z}$ is

$$\frac{-4}{z} = \frac{-4}{(1+(z-1))} = -4\left[1+(z-1)\right]^{-1} \quad (\text{Now } |z-1| < 1)$$

$$= -4\left[1-(z-1)+(z-1)^2 - \dots\right]$$

(ii) Laurent's series expansion for $\frac{3}{z+1}$ is.

$$\frac{3}{z+1} = \frac{3}{2+(z-1)} = \frac{3}{2\left(1+\frac{z-1}{2}\right)} \quad \left(\text{Now } \because |z-1| < 1, \ \left|\frac{z-1}{2}\right| < \frac{1}{2} < 1\right)$$

$$= \frac{3}{2}\left[1+\left(\frac{z-1}{2}\right)\right]^{-1}$$

$$= \frac{3}{2}\left[1-\frac{(z-1)}{2}+\frac{(z-1)^2}{4} - \dots\right]$$

(iii) Laurent's series expansion for $\frac{2}{z-2}$ is

$$\frac{2}{z-2} = \frac{-2}{1-(z-1)} = -2\left[1-(z-1)\right]^{-1}$$

$$= -2\left[1+(z-1)+(z-1)^2+\dots\right] \quad (\because |z-1| < 1)$$

6

$\therefore$ Laurent's series expansion of $f(z)$ is

$$f(z) = -4\left[1-(z-1)+(z-1)^2-\dots\right] + \frac{3}{2}\left[1-\frac{(z-1)}{2}+\frac{(z-1)^2}{4}-\dots\right] - 2\left[1+(z-1)+(z-1)^2+\dots\right]$$

$$= \left(-4+\frac{3}{2}-2\right) + \left(4(z-1) - \frac{3}{4}(z-1) - 2(z-1)\right)$$

$$+ \left(-4(z-1)^2 + \frac{3}{8}(z-1)^2 - 2(z-1)^2\right) + \left(4(z-1)^3 - \frac{3}{16}(z-1)^3 - 2(z-1)^3\right)$$

$$+ \cdots$$

$$= \frac{-9}{2} + \frac{5}{4}(z-1) - \frac{45}{8}(z-1)^2 + \cdots$$

(c) ann$(1;1,2)$

In this annulus, we have $1 < |z-1| < 2$

Now from (I)

$$f(z) = \frac{-4}{z} + \frac{3}{z+1} + \frac{2}{z-2}$$

(i) Laurent's series expansion for $\frac{-4}{z}$ is

$$\frac{-4}{z} = \frac{-4}{1+(z-1)} = \frac{-4}{(z-1)\left[1+\frac{1}{(z-1)}\right]} \qquad \left(\because |z-1|>1,\ \left|\frac{1}{z-1}\right|<1\right)$$

$$= \frac{-4}{(z-1)}\left[1+\left(\frac{1}{z-1}\right)\right]^{-1}$$

$$= \frac{-4}{(z-1)}\left[1-\left(\frac{1}{z-1}\right)+\left(\frac{1}{z-1}\right)^2-\cdots\right]$$

(ii) Laurent's series expansion for $3/(z+1)$ is

$$\frac{3}{z+1} = \frac{3}{2+(z-1)} = \frac{3}{2\left[1+\left(\frac{z-1}{2}\right)\right]} \quad \left(\because |z-1| < 2,\ \left|\frac{z-1}{2}\right| < \frac{1}{2} < 1 \right)$$

$$= \frac{3}{2}\left[1+\left(\frac{z-1}{2}\right)\right]^{-1} = \frac{3}{2}\left[1-\left(\frac{z-1}{2}\right)+\left(\frac{z-1}{2}\right)^2 - \cdots\right]$$

(iii) Laurent's series expansion for $2/(z-2)$ is

$$\frac{2}{z-2} = \frac{2}{(z-1)-1} = \frac{2}{(z-1)\left[1-\left(\frac{1}{z-1}\right)\right]} \quad \left(\because |z-1| > 1,\ \left|\frac{1}{z-1}\right| < 1 \right)$$

$$= \frac{2}{(z-1)}\left[1-\left(\frac{1}{z-1}\right)\right]^{-1}$$

$$= \frac{2}{(z-1)}\left[1+\left(\frac{1}{z-1}\right)+\left(\frac{1}{z-1}\right)^2+\cdots\right]$$

$\therefore$ Laurent's series expansion for $f(z)$ is

$$f(z) = \frac{-4}{(z-1)}\left[1-\frac{1}{(z-1)}+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\cdots\right]$$

$$+\frac{3}{2}\left[1-\frac{(z-1)}{2}+\frac{(z-1)^2}{4}-\frac{(z-1)^3}{8}+\cdots\right]$$

$$+\frac{2}{(z-1)}\left[1+\frac{1}{(z-1)}+\frac{1}{(z-1)^2}\right]$$

6

$$= \cdots - \frac{2}{(z-1)^3} + \frac{6}{(z-1)^2} - \frac{2}{(z-1)} + \frac{3}{2} - \frac{3}{4}(z-1) - \frac{3}{8}(z-1)^2$$

$$- \frac{3}{16}(z-1)^3 + \cdots$$

(d) $ann(i;\sqrt{2},\sqrt{5})$

In this annulus, $\sqrt{2} < |z-i| < \sqrt{5}$

From (I), $f(z) = \frac{-4}{z} + \frac{3}{z+1} + \frac{2}{z-2}$.

(i) Laurent's series expansion of $\frac{-4}{z}$ is

$$\frac{-4}{z} = \frac{-4}{(z-i)+i} = \frac{-4}{i\left[1+\left(\frac{z-i}{i}\right)\right]}$$ (Now $\sqrt{2}$

(Now $\sqrt{2} < |z-i| \Rightarrow \left|\frac{1}{z-i}\right| < \frac{1}{\sqrt{2}}$

i.e. $\left|\frac{i}{z-i}\right| < \frac{1}{\sqrt{2}}$ $\left(\because |i| = 1\right)$)

$$= \frac{-4}{i\left(\frac{z-i}{i}\right)\left[1+\frac{1}{\left[(z-i)/i\right]}\right]} = \frac{-4}{(z-i)\left(1+\left(\frac{i}{z-i}\right)\right)}$$

$$= \frac{-4}{z-i}\left[1+\left(\frac{i}{z-i}\right)\right]^{-1}$$

$$= \frac{-4}{z-i}\left[1-\left(\frac{i}{z-i}\right)+\left(\frac{i}{z-i}\right)^{2}-\ldots\right]$$

(ii) Laurent's series expansion for $3/_{1+z}$ is

$$\frac{3}{z+1} = \frac{3}{(z-i)+(1+i)}$$

Now $\sqrt{2} < |z-i|$

$\therefore \left|\frac{1}{z-i}\right| < \frac{1}{\sqrt{2}}$ $\therefore \left|\frac{1+i}{z-i}\right| < 1$ $\left(\because |1+i| = \sqrt{(1)^2+(1)^2} = \sqrt{2}\right)$

$$= \frac{3}{(z-i)\left[1+\left(\frac{1+i}{z-i}\right)\right]}$$

$$= \frac{3}{(z-i)}\left[1+\left(\frac{1+i}{z-i}\right)\right]^{-1}$$

$$= \frac{3}{(z-i)}\left[1-\left(\frac{1+i}{z-i}\right)+\left(\frac{1+i}{z-i}\right)^2-\ldots\right]$$

(iii) Laurent's series expansion for $\frac{2}{(z-2)}$ is

$$\frac{2}{z-2} = \frac{2}{(z-i)+(i-2)} = \frac{2}{(i-2)\left[1+\left(\frac{z-i}{i-2}\right)\right]}$$

$$\left(\text{Now } \because |z-i| < \sqrt{5} \qquad \left|\frac{z-i}{i-2}\right| < 1 \quad \left(\because \; |i-2| = \sqrt{(1)^2+(-2)^2} = \sqrt{5}\right)\right)$$

$$= \frac{2}{(i-2)}\left[1+\left(\frac{z-i}{i-2}\right)\right]^{-1}$$

$$= \frac{2}{(i-2)}\left[1-\left(\frac{z-i}{i-2}\right)+\left(\frac{z-i}{i-2}\right)^2-\ldots\right]$$

6

$\therefore$ Laurent's series expansion for $f(z)$ is

$$f(z) = \frac{2}{(i-2)} - \frac{1}{(z-i)} \; \frac{(i-3)}{(z-i)^2} + \frac{(4+6i)}{(z-i)^3} + \ldots$$

$$- \frac{2}{(i-2)^2}(z-i) + \frac{2}{(i-2)^3}(z-i)^3 - \ldots$$

$$= \ldots\ldots + \frac{8i+6}{(z-i)^4} + \frac{4+6i}{(z-i)^3} - \frac{(i-3)}{(z-i)^2} - \frac{1}{(z-i)} + \frac{2}{(i-2)}$$

$$- \frac{2}{(i-2)^2}(z-i) + \frac{2}{(i-2)^3}(z-i)^3 - \ldots\ldots\ldots$$

(e) $\operatorname{ann}(i; \sqrt{5}, \infty)$

In this annulus, $\sqrt{5} < |z-i| < \infty$

Thus $0 < \left|\frac{1}{z-i}\right| < \frac{1}{\sqrt{5}}$

(i) From (I), $f(z) = \frac{-4}{z} + \frac{3}{z+1} + \frac{2}{z-2}$

(i) Laurent's series expansion for $\frac{-4}{z}$ is

$$\frac{-4}{z} = \frac{-4}{(z-i)+i} = \frac{-4}{(z-i)\left[1+\left(\frac{i}{z-i}\right)\right]}$$

$\left(\text{Now } \because \left|\frac{1}{z-i}\right| < \frac{1}{\sqrt{5}},\ \left|\frac{i}{z-i}\right| < \frac{1}{\sqrt{5}} < 1 \quad (\because |i| = 1)\right)$

$$= \frac{-4}{(z-i)}\left[1+\left(\frac{i}{z-i}\right)\right]^{-1}$$

$$= \frac{-4}{(z-i)}\left[1 - \left(\frac{i}{z-i}\right) + \left(\frac{i}{z-i}\right)^2 - \ldots\ldots\right]$$

(ii) Laurent's series expansion for $\frac{3}{z+1}$ is

$$\frac{3}{z+1} = \frac{3}{(z-i)+(1+i)} = \frac{3}{(z-i)\left[1+\left(\frac{1+i}{z-i}\right)\right]}$$

$$\left(\text{Now } \left|\frac{1}{z-i}\right| < \frac{1}{\sqrt{5}} \Rightarrow \left|\frac{1+i}{z-i}\right| < \frac{\sqrt{2}}{\sqrt{5}} < 1\right)$$

$$= \frac{3}{(z-i)}\left[1+\left(\frac{1+i}{z-i}\right)\right]^{-1}$$

$$= \frac{3}{(z-i)}\left[1-\left(\frac{i+1}{z-i}\right)+\left(\frac{i+1}{z-i}\right)^2 - \cdots\cdots\right]$$

(iii) Laurent's series expansion for $\frac{2}{z-2}$ is

$$\frac{2}{z-2} = \frac{2}{(z-i)+(i-2)} = \frac{2}{(z-i)\left[1+\frac{i-2}{z-i}\right]}$$

$$\left(\text{Now } \left|\frac{1}{z-i}\right| < \frac{1}{\sqrt{5}} \Rightarrow \left|\frac{\sqrt{5}}{z-i}\right| < 1 \quad \text{i.e. } \left|\frac{i-2}{z-i}\right| < 1 \quad (\because |i-2| = \sqrt{5})\right)$$

$$= \frac{2}{(z-i)}\left[1+\left(\frac{i-2}{z-i}\right)\right]^{-1}$$

$$= \frac{2}{(z-i)}\left[1-\left(\frac{i-2}{z-i}\right)+\left(\frac{i-2}{z-i}\right)^2 - \cdots\cdots\right]$$

6

$\therefore$ Laurent's series expansion for $f(z)$ is

$$\frac{-4}{(z-i)}\left[1-\left(\frac{i}{z-i}\right)+\left(\frac{i}{z-i}\right)^2-\cdots\right] + \frac{3}{(z-i)}\left[1-\left(\frac{i+1}{z-i}\right)+\left(\frac{i+1}{z-i}\right)^2-\cdots\cdots\right]$$

$$+ \frac{2}{(z-i)}\left[1-\left(\frac{i-2}{z-i}\right)+\left(\frac{i-2}{z-i}\right)^2-\cdots\cdots\right]$$

$$= \left(\frac{-4+3+2}{z-i}\right. + \frac{4i-3(1+i)+2(2-i)}{(z-i)^2} + \frac{4+6i+2(-1+4-2i)}{(z-i)^3} + \cdots$$

$$= \left[\frac{1}{(z-i)} + \frac{(1-i)}{(z-i)^2} + \frac{(2i+10)}{(z-i)^3} - \cdots\cdots\right.$$

$$=(1+z^4+z^8+z^{12}+\cdots)(1+z^4+z^8+z^{12}+\cdots)\left(\left(\frac{2\pi}{z}\right)-\frac{(2\pi/z)^3}{3!}+\frac{(2\pi/z)^5}{5!}-\cdots\right)$$

$$=(1+2z^4+3z^8+4z^{12}+\cdots)\left(\frac{2\pi}{z}-\frac{(2\pi)^3}{3!\,z^3}+\frac{(2\pi)^5}{5!\,z^5}-\cdots\right)$$

$$=\frac{\left[(2\pi)+2\frac{(2\pi)^5}{5!}+3\frac{(2\pi)^9}{9!}+4\frac{(2\pi)^{13}}{13!}+\cdots\right]}{z}+\cdots$$

+3

$\therefore$ Residue of the function at the singularity

$$=(2\pi)+2\frac{(2\pi)^5}{5!}+3\frac{(2\pi)^9}{9!}+4\frac{(2\pi)^{13}}{13!}+\cdots$$

5] (a) $f(z)=\dfrac{z^3\sin(2z^3)}{\cos(z^4)-1}$

$$f(z)=\frac{z^3\left((2z^3)-\frac{(2z^3)^3}{3!}+\frac{(2z^3)^5}{5!}-\cdots\right)}{\left(1-\frac{(z^4)^2}{2!}+\frac{(z^4)^4}{4!}-\frac{(z^4)^6}{6!}+\cdots-1\right)}$$

$$=\frac{z^6\left(2-\frac{8z^6}{3!}+\frac{32z^{12}}{5!}-\cdots\right)}{-z^8\left(\frac{1}{2!}-\frac{z^8}{4!}+\frac{z^{16}}{6!}-\cdots\right)}$$

$$=\frac{-\left(2-\frac{8}{3!}z^6+\frac{32}{5!}z^{12}-\cdots\right)}{z^2\left(\frac{1}{2!}-\frac{z^8}{4!}+\frac{z^{16}}{6!}-\cdots\right)}$$

Thus $f(z)$ has pole of order 2 at $z=0$ because

$$\lim_{z\to 0} |f(z)| = \lim_{z\to 0} \frac{-\left(2 - \frac{8}{3!}z^6 + \frac{32}{5!}z^{12} - \cdots\right)}{z^2\left(\frac{1}{2!} - \frac{z^8}{4!} + \frac{z^{16}}{6!} - \cdots\right)}$$

$$= \frac{-\left(2 - \frac{8}{3!}(0)^6 + \frac{32}{5!}(0)^{12} - \cdots\right)}{(0)^2\left(\frac{1}{2!} - \frac{(0)^8}{4!} + \frac{(0)^{16}}{6!} - \cdots\right)}$$

$$= \infty$$

and it is pole of order 2 because $z^2 f(z)$ has a removable singularity at $z=0$ because

$$\lim_{z\to 0} z(z^2 f(z)) = \lim_{z\to 0} \frac{z^3\left(-2 + \frac{8}{3!}z^6 - \frac{32}{5!}z^{12} + \cdots\right)}{z^2\left(\frac{1}{2!} - \frac{z^8}{4!} + \frac{z^{16}}{6!} - \cdots\right)}$$

$$= \lim_{z\to 0} \frac{z\left(-2 + \frac{8}{3!}z^6 - \frac{32}{5!}z^{12} + \cdots\right)}{\left(\frac{1}{2!} - \frac{z^8}{4!} + \frac{z^{16}}{6!} - \cdots\right)}$$

$$= \frac{0(-2)}{\left(\frac{1}{2}\right)}$$

$$= 0$$

Hence we have pole of order 2 at $z=0$.

$$\text{Now } f(z) = \frac{-1\left(2 - \frac{8}{3!}z^6 + \frac{32}{5!}z^{12} - \cdots\right)}{\left(\frac{z^2}{2}\right)\left(1 - \frac{z^8}{12} + \frac{z^{16}}{360} - \cdots\right)}$$

$$= \frac{-2}{z^2}\left(2 - \frac{8}{3!}z^6 + \frac{32}{5!}z^{12} - \cdots\right)\left(1 - \left(\frac{z^8}{12} - \frac{z^{16}}{360} + \cdots\right)\right)^{-1}$$

$$= \frac{-2}{z^2}\left(2 - \frac{8}{3!}z^6 + \frac{32}{5!}z^{12} - \cdots\right)\left(1 + \left(\frac{z^8}{12} - \frac{z^{16}}{360} + \cdots\right) + \left(\frac{z^8}{12} - \frac{z^{16}}{360} + \cdots\right)^2 + \cdots\right)$$

Now if we multiplying above series and also with $-2/z^2$, we see that there is no term consisting of $\frac{1}{z}$.

Hence the coefficient of $1/z$ must be zero.

Hence residue of the function at $z=0$ is 0 (by definition)

(b) $f(z) = \frac{(z\sin z)^3}{(1-\cos(z^2))^2}$

Ans. $f(z)$ has pole of order 2 at $z=0$ because:-

$$f(z) = \frac{z^3 \sin^3 z}{(1-\cos(z^2))^2}$$

$$= \frac{z^3\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)^3}{\left[1 - \left(1 - \frac{z^4}{2!} + \frac{z^8}{4!} - \cdots\right)\right]^2}$$

$$= \frac{z^6\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)^3}{\left(\frac{z^4}{2!} - \frac{z^8}{4!} + \cdots\right)^2}$$

$$= \frac{z^6\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right)^3}{z^8\left(\frac{1}{2!}-\frac{z^4}{4!}+\cdots\right)^2}$$

$$= \frac{\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right)^3}{z^2\left(\frac{1}{2!}-\frac{z^4}{4!}+\cdots\right)^2}$$

$\therefore \lim_{z\to 0} |f(z)| = \infty$

Thus f has pole of order 2 at $z=0$ because $z^2 f(z)$ has a removable singularity at $z=0$ because

$$\lim_{z\to 0} z\left(z^2 f(z)\right) = \lim_{z\to 0} \frac{z^3\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right)^3}{z^2\left(\frac{1}{2!}-\frac{z^4}{4!}+\cdots\right)^2}$$

$$= \lim_{z\to 0} \frac{z\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right)^3}{\left(\frac{1}{2!}-\frac{z^4}{4!}+\cdots\right)^2}$$

$$= \frac{0\,(1)}{\left(\frac{1}{2!}\right)} = 0$$

To find residue of function at $z=0$:—

$$f(z) = \frac{\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right)^3}{z^2\left(\frac{1}{2!}-\frac{z^4}{4!}+\cdots\right)^2}$$

$$\frac{\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right)^3}{\frac{z^2}{4}\left(1-\frac{z^4}{12}+\frac{z^8}{360}-\cdots\right)^2}$$

$$=\frac{\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right)^3}{\frac{z^2}{4}\left(1-\frac{z^4}{12}+\frac{z^8}{360}-\cdots\right)\left(1-\frac{z^4}{12}+\frac{z^8}{360}-\cdots\right)}$$

$$=\frac{4}{z^2}\cdot\frac{\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right)^3}{\left(1-\frac{z^4}{6}+\frac{9z^8}{720}-\cdots\right)}$$

$$=\frac{4}{z^2}\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right)^3\left(1-\left(\frac{z^4}{6}-\frac{9z^8}{720}+\cdots\right)\right)^{-1}$$

$$=\frac{4}{z^2}\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right)^3\left(1+\left(\frac{z^4}{6}-\frac{9z^8}{720}+\cdots\right)+\left(\frac{z^4}{6}-\frac{9z^8}{720}+\cdots\right)^2+\cdots\right)$$

Now ~~the~~ ~~e~~ if we multiplying all the above terms, we see that, we have constant term, positive powers of z and and a term containing $\frac{1}{z^2}$.

~~So~~ There is no term consisting of $\frac{1}{z}$.

Thus coefficient of $\frac{1}{z}$ must be zero.

Hence residue of function f at $z=0$ is zero.