MA 502 - Tutorial 8

(1) Note that either $f(a)=0$ or $f(a)\neq 0$. If $f(a)=0$, we are done. If not, then $0<|f(a)|\leq |f(z)|\ \forall z\in G$.
Let $g: G\to \mathbb{C}$ be defined by $g(z)=\dfrac{1}{f(z)}$ for $z\in G$.
Note that g is analytic on G.
Then we have $|g(z)|\leq |g(a)|\ \forall z\in G$. By the 1st version of the Max. Mod. Thm, we have g (and hence f) is a constant.

(2) If f has a zero in G, we are done. If not, then $g: G\to \mathbb{C}$ defined by $g(z)=\dfrac{1}{f(z)}$, $z\in G$, is a non-constant analytic function on G. By the 2nd version of the Max. Mod. Thm,

$$\max\{|g(z)| : z\in \bar{G}\} = \max\{|g(z)| : z\in \partial G\}.$$

Thus $|g|$ assumes its maximum value of ∂G. In other words, $|f|$ assumes its minimum value on ∂G.

(3) If f has a zero in G, then we are done. If not, then by problem (2) above as well as from the 2nd version of the max. mod. thm,

$$\min_{z\in\bar{G}}|f(z)| = \min_{z\in\partial G}|f(z)| = c = \max_{z\in\partial G}|f(z)| = \max_{z\in\bar{G}}|f(z)|.$$

$\Rightarrow f$ must be constant.

(4) Case 1: Suppose f & g have no zeros on $\partial B(0;R)$.
Then $\dfrac{f}{g}$ is analytic on $\bar{B}(0;R)$ with $\left|\dfrac{f(z)}{g(z)}\right|=1$ on $\partial B(0;R)$.
Note that f/g also has no zeros in G. Then by prob (3) above, f/g is constant, say, λ. Thus
$f=\lambda g$ with $|\lambda|=1$.

Case 2: f and g have zeros on $\partial B(0;R)$. Note that $|f(z)| = |g(z)|$ on $\partial B(0;R)$ implies they have same zeros on $\partial B(0;R)$.

Since f and g are analytic on $\overline{B}(0;R)$, their zeros are isolated, and each zero is of finite multiplicity. Suppose the zeros that are distinct be $z_1, z_2, \ldots, z_n$.
If z_1 is a zero of f of multiplicity k and of g with multiplicity k', then

$$\left.\begin{aligned} f(z) &= (z-z_1)^k f_1(z) \\ g(z) &= (z-z_1)^{k'} g_1(z), \end{aligned}\right\} \begin{aligned} &f_1, g_1 \text{ analytic on } \overline{B}(0;R) \\ &f_1(z_1) \neq 0,\ g_1(z_1) \neq 0 \end{aligned}$$

$$\Rightarrow |z-z_1|^k |f_1(z)| = |z-z_1|^{k'} |g_1(z)| \ \forall z \ni |z| = R.$$

Suppose $k \neq k'$. w.l.g., let $k > k'$.
Then $|z-z_1|^{k-k'} |f_1(z)| = |g_1(z)|$
Now let $\underset{(|z|=R)}{z} \to z_1$. This implies $|g_1(z)| = 0$
i.e. $g_1(z) = 0$
contradiction to

Hence $k = k'$. Similarly any zero in $\partial B(0;R)$ has the same multiplicities for f & g.

Let k_i be the multiplicity of z_i. Then define

$$\tilde{f}(z) := \frac{f(z)}{(z-z_1)^{k_1} \cdots (z-z_n)^{k_n}} \quad \& \quad \tilde{g}(z) := \frac{g(z)}{(z-z_1)^{k_1} \cdots (z-z_n)^{k_n}}.$$

So $\tilde{f}$ & $\tilde{g}$ are analytic with no zeros on $\overline{B}(0;R)$ and $|\tilde{f}(z)| = |\tilde{g}(z)|$ for $|z| = R$. Hence by case 1, $\exists\, \lambda \ni \tilde{f} = \lambda \tilde{g}$ and $|\lambda| = 1$.

$$\Downarrow$$

$$f = \lambda g$$

$\boxtimes$

(4) Suppose $r<s<R$. Let $g(z)=e^{f(z)}$. Then $|g(z)|=e^{Re(f(z))}$.

Also, g is analytic in $B(0;R)$.

By 2nd version of Max. mod. Thm; applied to g and with the bounded open set $B(0;s)$, we have

$$\max\{e^{Re(f(z))} : |z|=s\} = \max\{e^{Re(f(z))} : |z|\le s\}.$$

However, if $|z|=r$, then $|z|\le s$. Hence

$$\max\{e^{Re(f(z))} : |z|\le s\} \ge \max\{e^{Re(f(z))} : |z|=r\}.$$

$\Rightarrow A(r)\le A(s)$.

Now suppose $A(s)=A(r)$ for some $r<s<R$.

Then $\exists\, z_0$ with $|z_0|=r \ni |g(z_0)|\ge|g(z)|\ \forall z \ni |z|<s$.

By 1st version of Max. mod. Thm;

f is constant.

AV