MA 502 : TUTORIAL 9

① Let (z,w) denote an ordered pair of complex numbers. The distance between two such pairs is given by $|(z_1,w_1)-(z_2,w_2)| = \sqrt{(z_1-z_2)^2+(w_1-w_2)^2}$.
(Metric of $\mathbb{C}^2 = \mathbb{C}\times\mathbb{C}$)

Ⓐ CONTINUITY OF $\varphi(z,w)$:

The main part is to show that for $a \in G$ fixed,

$$\boxed{\lim_{(z,w)\to(a,a)} \varphi(z,w) = \varphi(a,a) = f'(a)} . \quad \text{— } (1)$$

This is because when $z \neq w$, continuity is clear: $f(z)-f(w)$ is continuous (being analytic) and $z-w$ is continuous (with $z\neq w$), so their quotient is continuous too. — ②

To prove ①, let $\varepsilon > 0$ be given.
Since f being analytic implies f' is continuous, $\exists\, \delta > 0 \ni |f'(\xi)-f'(a)| < \varepsilon$ whenever $\xi \in B(a;\delta) \subseteq G$.
We claim that the same δ works for proving ①, i.e., we show that $|(z,w)-(a,a)| < \delta$ implies

$$|\varphi(z,w) - \varphi(a,a)| < \varepsilon .$$

To that end, note that by the fundamental theorem of Calculus,

$$\boxed{\varphi(z,w) = \int_0^1 f'(tz+(1-t)w)\,dt} \quad \text{— } (3)$$

$$\left(\because \int_0^1 f'(tz+(1-t)w)\,dt = \left[\frac{f(tz+(1-t)w)}{z-w}\right]_0^1 = \frac{f(z)-f(w)}{z-w}\right.$$

whenever $z \neq w$, and

when $z=w$, $\int_0^1 f'(tz+(1-t)w)\,dt = \int_0^1 f'(w)\,dt = f'(w)$.)

Then the next step is to note that

$$|z-a| \leq \sqrt{|z-a|^2+|w-a|^2} = |(z,w)-(a,a)| < \delta$$

& similarly, $|w-a|<\delta$.

Since open balls are convex, and $z \in B(a;\delta)$ & $w \in B(a;\delta)$, we deduce that

$$tz+(1-t)w \in B(a;\delta) \quad \text{for } 0 \leq t \leq 1.$$

$$\Rightarrow |(tz+(1-t)w)-a| < \delta$$

So that $|f'(tz+(1-t)w)-f'(a)| < \varepsilon$ —— (4)

Hence from (3) and (4),

$$|\varphi(z,w)-\varphi(a,a)|$$

$$= \left| \int_0^1 f'(tz+(1-t)w)\,dt - \int_0^1 f'(a)\,dt \right|$$

$$= \left| \int_0^1 \left(f'(tz+(1-t)w) - f'(a)\right) dt \right|$$

$$\leq \int_0^1 |f'(tz+(1-t)w) - f'(a)|\,dt$$

$$< \int_0^1 \varepsilon\,dt = \varepsilon.$$

This proves (1), and hence from (1) & (2), we see that $\varphi(z,w)$ is continuous on $G \times G$.

(B) ANALYTICITY OF $z \mapsto \varphi(z,w)$ for each fixed w

Fix $w=b \in G$. and let $g(z) = \varphi(z,b)$.

For $z \neq b$,

$$g(z) = \varphi(z,b) = \frac{f(z)-f(b)}{z-b}$$

$$= \frac{\sum_{n=0}^{\infty} a_n (z-b)^n - f(b)}{z-b}$$ (∵ f is analytic on G, we expand f as a power series around b)

$$= \frac{\sum_{n=0}^{\infty} a_n (z-b)^n - a_0}{z-b} = \sum_{n=1}^{\infty} a_n (z-b)^{n-1} \quad \text{—— (5)}$$

Also for $z = b$, $g(z) = \varphi(b,b) = f'(b)$, which is also given by (5). (since $a_n = \frac{f^{(n)}(b)}{n!}$ implies, in particular, $a_1 = f'(b)$).

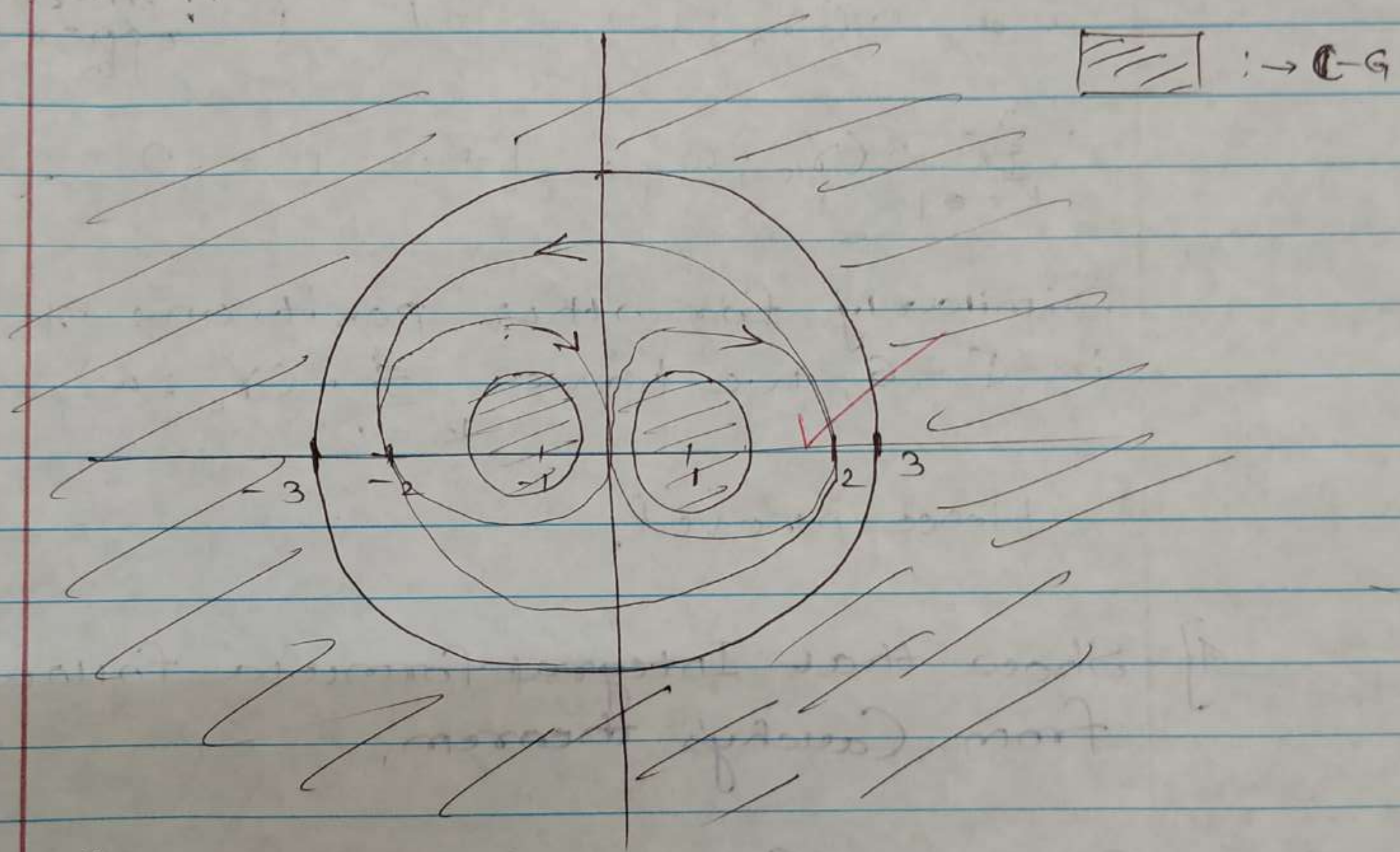
Since power series are analytic functions, we conclude that g is analytic on G.

$\boxtimes$

3] Let $B_\pm = \bar{B}(\pm 1; \frac{1}{2})$, $G = B(0,3) \setminus (B_+ \cup B_-)$.

Let $\gamma_1, \gamma_2, \gamma_3$ be curves whose traces are $|z-1| = 1$, $|z+1| = 1$, and $|z| = 2$ respectively. Give $\gamma_1, \gamma_2, \gamma_3$ orientations such that $n(\gamma_1; w) + n(\gamma_2; w) + n(\gamma_3; w) = 0$ $\forall w$ in $\mathbb{C} - G$,

Ans.



Since we want
$\sum_{k=1}^{3} n(\gamma_k; w) = 0$, we must have,

γ_1, γ_2 of positive orientation and
γ_3 of negative orientation

<u>OR</u>

γ_1, γ_2 are of negative orientation and
γ_3 is of positive orientation.

This is because $\{\gamma_3\} \supset \{\gamma_1\} \cup \{\gamma_2\}$ and $\mathbb{C} - G$ is ~~as~~ the shaded region as shown. So if $w \in \mathbb{C} - G$ belongs to $\{\gamma_1\}$ (say), then γ_1 being a circle winds its once. $\therefore n(\gamma_1 ; w) = 1$

But $n(\gamma_2 ; w) = 0$

and $n(\gamma_3 ; w) = -1$ ($\because$ since of opposite orientation)

$$\therefore \sum_{k=1}^{3} n(\gamma_k ; w) = 1 + 0 - 1 = 0$$

Similarly for other positions of w in $\mathbb{C} - G$, we have $\sum_{k=1}^{3} n(\gamma_k ; w) = 0$

Hence proved.

4] Show that Integral formula follows from Cauchy's theorem.

Given :- G is an open subset of $\mathbb{C}$ and $f : G \to \mathbb{C}$ is analytic. If $\gamma_1, \gamma_2, \dots, \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1 ; w) + \dots + n(\gamma_m ; w) = 0 \ \forall w \in \mathbb{C} - G$, then ~~for a in $G - \{\gamma\}$,~~

$$\sum_{k=1}^{m} \int_{\gamma_k} f = 0.$$

To prove :-

$$\sum_{k=1}^{m} n(\gamma_k; a) f(a) = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(z)\, dz}{(z-a)}$$

Proof :- Construct a function g as follows :-

$$g(z) = \begin{cases} \dfrac{f(z) - f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

by ~~part (a)~~ problem (1), $g(z)$ is continuous and also analytic.

Now in the given hypothesis, take $\phi = g$.
So by Cauchy's theorem

$$\sum_{k=1}^{m} \int_{\gamma_k} g(z)\, dz = 0.$$

But here since $a \notin \{\gamma_k\}$ $\forall\, k = 1 \ldots m$
we never have to deal with second definition of g i.e. $g(z) = f'(a)$ when $z = a$
because $\because$ we are integrating over γ_k, $z \in \gamma_k$ and so, $z \neq a$ otherwise $a \in \gamma_k$ which is false.
Hence while proving Cauchy's integral formula, $g(z) = \dfrac{f(z) - f(a)}{z-a}$

6] Let f be analytic on $D = B(0;1)$ and suppose that $|f(z)| \le 1$ for $|z| < 1$. Show that $|f'(0)| \le 1$

Proof:- f is analytic on $B(0;1)$, Also f is bounded i.e. $|f(z)| \le 1$ for $z \in B(0;1)$.
Hence by Cauchy's estimate,

$$|f^{(n)}(0)| \le \frac{n!M}{R^n}$$

But here $n = 1$ and $M = 1$, $R = 1$

$$\therefore |f'(0)| \le \frac{1!(1)}{1^{(1)}}$$

$$\therefore |f'(0)| \le 1$$

Hence proved.

7] $\gamma(t) = 1 + e^{it}$ for $0 \le t \le 2\pi$. Find $\int_\gamma \left(\frac{z}{z-1}\right)^n dz$ for all positive integers n.

Ans. $\left(\frac{z}{z-1}\right)^n$ has singularity at $z = 1$

and $\gamma(t)$ contains that singularity

~~So~~ Also $\gamma(t)$ is a circle, so it winds around $z=1$ once. So $n(\gamma;1)=1$

Now from Cauchy's theorem,

$$f^{(k)}(a)\, n(\gamma;a) = \frac{k!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{k+1}}\,dz.$$

Now here $k=n-1$, $a=1$, $f(z)=z^n$

$$\therefore f^{(n-1)}(z) = n(n-1)\cdots 2\cdot 1\, z^{n-(n-1)}$$
$$= n!\, z^n$$

$$\therefore f^{(n-1)}(1) = n!\,(1)^n = n!$$

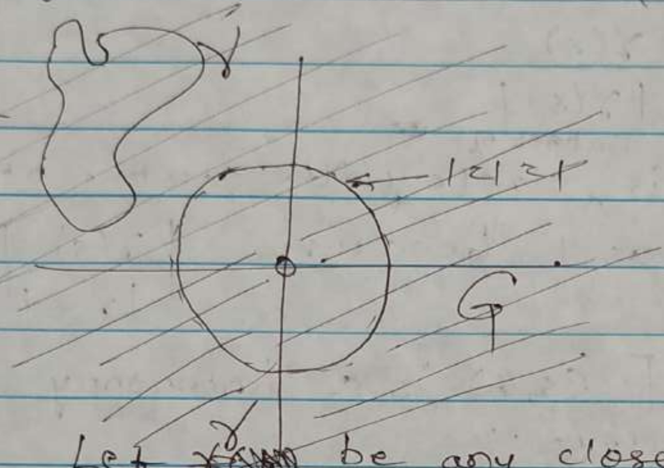
$$\therefore n!\,(1) = \frac{(n-1)!}{2\pi i}\int_\gamma \left(\frac{z}{(z-1)}\right)^n dz$$

$$\therefore \int_\gamma \left(\frac{z}{z-1}\right)^n dz = \frac{2\pi i\, n!}{(n-1)!}$$
$$= 2\pi i n$$
$$= 2\pi n i$$

$$\therefore \int_\gamma \left(\frac{z}{z-1}\right)^n dz = 2\pi n i$$

4] Let $G = \mathbb{C} - \{0\}$. Show that every closed curve in G is homotopic to a closed curve whose traced is contained in $\{z : |z| = 1\}$.

Proof:-



Let γ ~~γ(s)~~ be any closed curve as shown.
We construct ~~a~~ a ~~co~~ function $\Gamma(s,t)$ from $[0,1] \times [0,1]$ to G such ~~that~~

$$\Gamma(s,t) = \frac{\gamma(s)}{(1-t) + t|\gamma(s)|}$$

$\Gamma(0,t) = \Gamma(1,t)$ since $\gamma(0) = \gamma(1)$

Now since $\gamma : [0,1] \to \mathbb{C}$ is a continuous function and $(1-t) + t|\gamma(s)|$ is also continuous ($\because |\gamma(s)|$ is continuous), $\Gamma(s,t)$ is also a continuous function.

Now when $t = 0$

$$\Gamma(s,0) = \frac{\gamma(s)}{(1-0) + 0(|\gamma(s)|)} = \gamma(s)$$

Thus we get the original curve γ when $t = 1$

When $t = 1$,

$$\Gamma(s, 1) = \frac{\gamma(s)}{(1-1) + |(\gamma(s)|}$$

$$= \frac{\gamma(s)}{|\gamma(s)|}$$

Now this is subset of circle with center at origin and radius as 1. because $\left|\frac{\gamma(s)}{|\gamma(s)|}\right| = \frac{|\gamma(s)|}{|\gamma(s)|}$

Hence ~~ex~~ $\Gamma(s,t)$ is a homotopy between γ and $\{z : |z| = 1\}$

Hence every closed curve in G is homotopic to a closed curve whose trace is contained in $\{z : |z| = 1\}$.

Hence proved.