MA 502 - Tutorial 11 (Complex Analysis)

1. Let $p(z)$ be a polynomial of degree n and let $R > 0$ be sufficiently large so that p never vanishes in $\{z : |z| > R\}$. If $\gamma(t) = Re^{it}, 0 \le t \le 2\pi$, show that $\displaystyle\int_\gamma \frac{p'(z)}{p(z)}\, dz = 2\pi i n$.

Hint: You may make use of the fact that if $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, then as $|z| \to \infty$, for any $\epsilon > 0$,

$$(1-\epsilon)|a_n||z|^n \le |p(z)| \le (1+\epsilon)|a_n||z|^n.$$

2. Each of the following functions f has an isolated singularity at $z = 0$. Determine its nature; if it is a removable singularity define $f(0)$ so that f is analytic at $z = 0$; if it is a pole find the singular part; if it is an essential singularity, say why.

(a) $f(z) = \dfrac{\sin(z)}{z}$ (b) $f(z) = \dfrac{\cos(z)}{z}$ (c) $f(z) = \dfrac{\cos(z)-1}{z}$

(d) $f(z) = \dfrac{\log(z+1)}{z^2}$ (e) $f(z) = z^n \sin\left(\dfrac{1}{z}\right)$.

3. Find the Laurent series expansion of $f(z) = \dfrac{8+z^2}{z^3 - z^2 - 2z}$ on:

(a) $\text{ann}(0; 1, 2)$ (b) $\text{ann}(1; 0, 1)$ (c) $\text{ann}(1; 1, 2)$

(d) $\text{ann}(i; \sqrt{2}, \sqrt{5})$ (e) $\text{ann}(i; \sqrt{5}, \infty)$.

4. Determine the order of the pole at $z = 0$ of the following functions:

(a)$f(z) = \dfrac{z^3 \sin(2z^3)}{\cos(z^4) - 1}$ (b) $f(z) = \dfrac{(z\sin(z))^3}{(1-\cos(z^2))^2}$.