**MA 502 - Tutorial 12 (Complex Analysis)**

1. Evaluate the following integrals:

(a) $\displaystyle\int_0^\infty \frac{x^2}{x^4+x^2+1}\,dx$ (b) $\displaystyle\int_0^\pi \frac{\cos(2\theta)\,d\theta}{1-2a\cos(\theta)+a^2}$, where $a^2<1$.

2. Suppose $f(z)=g(z)/h(z)$, where $g$ and $h$ are analytic in a neighborhood of $a$, $g(a)\neq 0$ and $h(z)$ has a simple zero at $z=a$. Then prove that

$$\operatorname{Res}(f;a)=\frac{g(a)}{h'(a)}.$$

3. Suppose that $f$ has an essential singularity theorem at $z=a$. Prove the following strengthened version of the Casorati-Weierstrass theorem:

If $c\in\mathbb{C}$ and $\epsilon>0$ are given, then for each $\delta>0$, there is a number $\alpha$, with $|c-\alpha|<\epsilon$, such that $f(z)=\alpha$ has infinitely many solutions in $B(a;\delta)$.

(**Hint:** (a) You may assume the Baire category theorem, namely, in a complete metric space, if $\{U_n\}_{n=1}^\infty$ is a collection of open dense sets in $X$, then $\cap_{n=1}^\infty U_n$ is dense in $X$.

(b) You can also assume the fact that if $A$ is a set dense in $X$, then $A$ intersects every non-empty open set in $X$.)

4. Let $\gamma$ be the rectangular path $[n+\frac{1}{2}+ni, -n-\frac{1}{2}+ni, -n-\frac{1}{2}-ni, n+\frac{1}{2}-ni, n+\frac{1}{2}+ni]$ and evaluate the integral $\displaystyle\int_\gamma \frac{\pi\cot(\pi z)}{(z+a)^2}\,dz$ for $a$ not equal to an integer. Show that $\lim_{n\to\infty}\displaystyle\int_\gamma \frac{\pi\cot(\pi z)}{(z+a)^2}\,dz=0$ and, by using the first part, deduce that

$$\frac{\pi^2}{\sin^2(\pi a)}=\sum_{n=-\infty}^{\infty}\frac{1}{(a+n)^2}.$$

(Hint: Use the fact that for $z=x+iy$, $|\cos(z)|^2=\cos^2(x)+\sinh^2(y)$ and $|\sin(z)|^2=\sin^2(x)+\sinh^2(y)$ to show that $|\cot(\pi z)|\leq 2$ for $z$ on $\gamma$ if $n$ is sufficiently large.)