

Problem 4 from Tutorial 5

* Does there exist a non-empty perfect set in \mathbb{R} that contains no rational numbers?

Ans. Yes!

An example due to Julia Head, Eric Murphy & Brenton Calloway

Let $\{\gamma_i\}_{i=1}^{\infty}$ be the sequence of rationals

Pick $\varepsilon_i = k_i \sqrt{2}$, where $k_i \in \mathbb{Q}^+$
 $\Rightarrow \varepsilon_i \in \mathbb{R} \setminus \mathbb{Q}$.

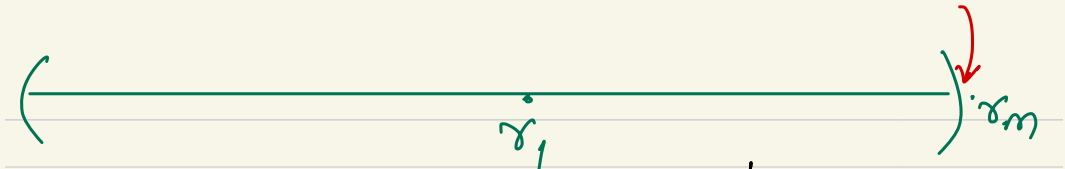
Consider $I_1 = (\gamma_1 - \varepsilon_1, \gamma_1 + \varepsilon_1)$

I_1 contains infinitely many rationals γ_i
But there are infinitely many $\gamma_i \notin I_1$.

Pick the smallest such index $i \ni \gamma_i \notin I_1$, say m . (Thus $\gamma_i \notin I_1$ if $i \geq m$).

Let $\varepsilon_m = k_m \sqrt{2}$, where $k_m \in \mathbb{Q}^+$ and
 $\varepsilon_m < \text{distance of } \gamma_m \text{ from } I_1$

Having constructed I_1, I_2, \dots, I_{n-1} , let γ_m be the rational with the smallest index m s.t. $\gamma_m \notin I_1 \cup I_2 \cup I_3 \cup \dots \cup I_{n-1}$



and let $\varepsilon_n = k_n \sqrt{2}$, $k_n \in \mathbb{Q}^+$ and s.t.

$$I_n \cap \overline{I_1 \cup I_2 \cup \dots \cup I_{n-1}} = \emptyset.$$

First, $\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_n \quad \text{---} \quad (*)$

$\bigcup_{n=1}^{\infty} I_n$ is open. $\text{---} \quad (1)$

Let $P = \mathbb{R} \setminus \left(\bigcup_{n=1}^{\infty} I_n \right)$.

- P is closed (from (1))
- P contains no rationals (from $(*)$)
- Claim: Every point of P is a limit point of P .

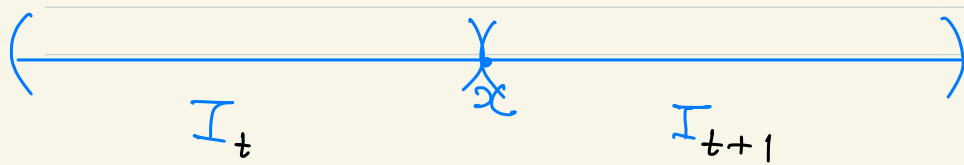
Proof: (by contradiction)

Suppose $\exists x \in P$ but $x \notin P'$.

$$\Rightarrow \exists \delta > 0 \exists N_\delta(x) \cap P = \{x\}.$$

$$\Rightarrow N_\delta(x) \setminus \{x\} \subset \bigcup_{n=1}^{\infty} I_n$$

$\Rightarrow x$ must be the open end point of 2 consecutive intervals



$\Rightarrow \exists i, j \in \mathbb{N}$ s.t.

$$x = x_i - \varepsilon_i = x_j + \varepsilon_j$$

$$\Rightarrow x_i - x_j = \varepsilon_i + \varepsilon_j = \sqrt{2} (k_i + k_j)$$

\mathbb{Q} $\mathbb{R} \setminus \mathbb{Q}$

\Rightarrow contradiction

Hence $x \in P$.

$\Rightarrow P$ is perfect.