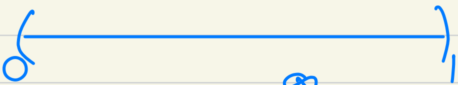
18/9/2020

MA 509 - REAL ANALYSIS - LECT. 15

COMPACT SETS



$G_n = \left(\frac{1}{n}, 1\right), n \in \mathbb{N}$

Then $(0,1) \subset \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 1\right)$. Let $x \in (0,1)$
$\exists\, m \in \mathbb{N} \ni \frac{1}{m} < x \Rightarrow x \in G_m$.
$\Rightarrow \{G_n : n \in \mathbb{N}\}$ forms an open cover of $(0,1)$.

Defn. Open cover : An open cover of a set E in a metric space X is a collection $\{G_\alpha\}$ of open subsets of $X \ni E \subset \bigcup_\alpha G_\alpha$.

Defn. A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

This means if $\{G_\alpha\}$ is an open cover of K, that is, $K \subset \bigcup_\alpha G_\alpha$. Then $\exists$ finitely many $\alpha_1, \alpha_2, \cdots \alpha_n \ni$

$$K \subset \bigcup_{i=1}^{n} G_{\alpha_i}$$

Eg. A finite set is always compact.

Thm. 2.18 Suppose $K \subset Y \subset X$. Then K is compact relative to X iff K is compact relative to Y.

Proof: "$\Rightarrow$" Suppose K is compact relative to X. Let $\{V_\alpha\}$ be an open cover of K consisting of open sets in Y.

Claim: $K \subset \bigcup_{i=1}^{n} V_{\alpha_i}$ for some V_{α_i}'s in $\{V_\alpha\}$.

Note that by the previous theorem, $\exists \{G_\alpha\}$ — a collection of open sets in X — such that

$$V_\alpha = Y \cap G_\alpha \text{ for each } \alpha.$$

Since K is compact relative to X and $\{G_\alpha\}$ is an open cover of K $\left(\because K \subset \bigcup_\alpha V_\alpha \subset \bigcup_\alpha G_\alpha\right)$ there exists finitely many indices, say, $\alpha_1, \alpha_2, \ldots, \alpha_n \ni$

$$K \subset \bigcup_{i=1}^{n} G_{\alpha_i}$$

Since $K \subset Y$,

$$\begin{aligned} K &\subset \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cap Y \\ &= \bigcup_{i=1}^{n} (G_{\alpha_i} \cap Y) \\ &= \bigcup_{i=1}^{n} V_{\alpha_i} \end{aligned}$$

$\Rightarrow K$ is compact relative to Y.

"$\Leftarrow$" Suppose K is compact relative to Y. Let $\{G_\alpha\}$ be an open cover of K consisting of open sets in X.

Let $V_\alpha = Y \cap G_\alpha$

Now $K \subset \bigcup_\alpha V_\alpha$ $\left(\because K \subset Y \;\&\; \bigcup_\alpha G_\alpha \text{ is an open cover of } K \text{ and since } Y \cap \left(\bigcup_\alpha G_\alpha\right) = \bigcup_\alpha Y \cap G_\alpha = \bigcup_\alpha V_\alpha\right)$

$\exists$ finitely many indices $\alpha_1, \cdots, \alpha_n \ni$
$$K \subset \bigcup_{i=1}^{n} V_{\alpha_i}.$$

But $V_{\alpha_i} \subset G_{\alpha_i}$ $\forall\, i \ni 1 \leq i \leq n.$

$$\Rightarrow K \subset \bigcup_{i=1}^{n} G_{\alpha_i}.$$

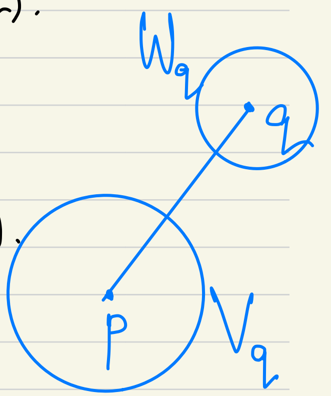
$\Rightarrow$ K is compact relative to X.

Thm. 2.19 Compact subsets of metric spaces are closed.

Proof: Let K be a compact subset of a metric space X.

Claim: K^c is open.

Let $p \in K^c$. We show $\exists$ nbhd V of $p \ni V \subset K^c$. To that end, let $q \in K$. Then $d(p,q) > 0$. Then consider nbhds V_q and W_q of p and q respectively of radius $< \frac{1}{2} d(p,q)$.



Let $\bigcup_{q \in K} W_q$ be an open cover of K.

Then K compact implies $\exists$ a finite sub-cover $\bigcup_{i=1}^{n} W_{q_i} \ni K \subset \bigcup_{i=1}^{n} W_{q_i} =: W$.

Take corresponding nbhds $V_{q_1}, V_{q_2}, V_{q_3}, \cdots, V_{q_n}$ of p & let $V = \bigcap_{i=1}^{n} V_{q_i}$.

First, V is open. (finite intersection of open sets)
By construction, $V \cap W = \phi$.
Also, $p \in V$ and $K \subset W$.
$\Rightarrow$ The nbhd V of p is contained in K^c.
$\Rightarrow$ p is an interior pt. of K^c.
$\Rightarrow$ K^c is open, so K is closed. ▨

Thm. 2.20 Closed subsets of compact sets are compact.

Proof: Let $F \subset K \subset X$, where F is closed (relative) to the metric space X, and K is compact. Then, let $\{V_\alpha\}$ be an open cover of F.
Then $\{V_\alpha\} \cup \underset{\text{(open)}}{F^c}$ is an open cover of K.

- By compactness of K, $\exists$ a finite subcover of $\{V_\alpha\} \cup F^c$ which covers K (and hence F).

K compact $\Rightarrow \exists\, V_{\alpha_i}$, $1 \le i \le n$, $\ni$ $K \subset \left(\bigcup_{i=1}^{n} V_{\alpha_i}\right) \cup F^c$.

- If F^c is a member of this finite subcover, remove it; still we get a finite subcover of F.

$\Rightarrow$ F is compact. ▨

Cor. 2.21 If F is closed and K is compact, then $F \cap K$ is compact.

Proof: K compact $\Rightarrow$ K closed
$\Rightarrow$ $F \cap K$ closed & thus a closed subset

of compact set K, and hence compact. ∎