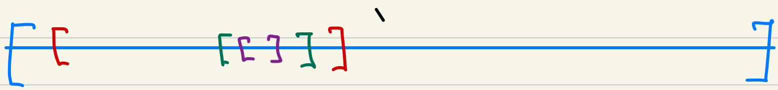
22/9/20

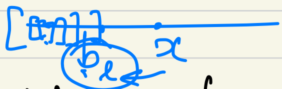
MA 509 - REAL ANALYSIS - LECT. 17

Thm. 2.24 If $\{I_n\}$ is a sequence of closed intervals in $\mathbb{R}^1$ s.t. $I_n \supset I_{n+1}$, $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \phi$.



Proof: Let $I_n = [a_n, b_n]$, $n \in \mathbb{N}$.
Let $E := \{a_n : a_n$ is the left end-point of $I_n\}$.
Then E is non-empty and bounded above by b_1 (Why?)

So $\sup(E)$ exists in $\mathbb{R}$, say, x.
Let $m, n \in \mathbb{N}$. Then
$$a_n \leq a_{m+n} \leq b_{m+n} \leq b_n$$
$\Rightarrow$ $x \leq b_m$ $\forall m \in \mathbb{N}$, for otherwise, $x \neq \sup(E)$
(Explain why?)



Since $a_m \leq x$ $\forall m \in \mathbb{N}$, we have $x \in I_m$ $\forall m \in \mathbb{N}$
$\Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \phi$. ∎

Thm. 2.25 Let $k \in \mathbb{N}$. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$ $(n \in \mathbb{N})$, then $\bigcap_{n=1}^{\infty} I_n \neq \phi$.

Proof: Note that for every $n \in \mathbb{N}$,
$$I_n = \{\bar{x} = (x_1, \ldots, x_k) : a_{n,j} \leq x_j \leq b_{n,j}, 1 \leq j \leq k\}.$$

1-cell : a [——] b
2-cell : $[a, b] \times [c, d]$

Note that
$I_{1,j} \supset I_{2,j} \supset I_{3,j} \supset \cdots$
$I_{1,2} \supset I_{2,2} \supset I_{3,2} \supset \cdots$

Let $I_{n,j} := [a_{n,j}, b_{n,j}]$

Note that $I_{n,j}$ satisfies the hypotheses of Thm. 2.24 for each j. Hence $\exists$ real numbers x_j^*, $1 \leq j \leq k$, s.t.

$$a_{n,j} \leq x_j^* \leq b_{n,j} \qquad 1 \leq j \leq k, \ n \in \mathbb{N}.$$

Now let $\bar{x}^* = (x_1^*, \ldots, x_k^*)$.
Then $\bar{x}^* \in I_n \ \forall n \in \mathbb{N}$. Hence $\bigcap_{n=1}^{\infty} I_n \neq \phi$. ∎

<u>Thm. 2.26</u> Every k-cell is compact.

<u>Proof:</u> Let I be a k-cell. Then

$$I = \{\bar{x} = (x_1, \ldots, x_k) : a_j \leq x_j \leq b_j, \ 1 \leq j \leq k\}.$$

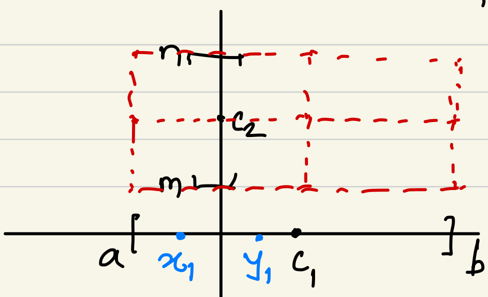
Let $\delta = \sqrt{\sum_{j=1}^{k} (b_j - a_j)^2}$.

Then if $\bar{x} \in I$, $\bar{y} \in I$, then $|\bar{x} - \bar{y}| \leq \delta$. (Why?)

If $\bar{x} = (x_1, \ldots, x_k)$ & $\bar{y} = (y_1, \ldots, y_k)$, then

$$|\bar{x} - \bar{y}| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_k - x_k)^2} \leq \sqrt{\sum_{j=1}^{k} (b_j - a_j)^2} = \delta$$

Suppose, by means of contradiction, $\exists$ an open cover $\{G_\alpha\}$ of I having no finite subcover for I.
Let $c_j := (a_j + b_j)/2$. Then the intervals $[a_j, c_j]$ & $[c_j, b_j]$ determine 2^k k-cells Q_i, whose union is I.



$a \leq x \leq b$
$c \leq y \leq d$

Then at least one of these sets Q_i, say I_1, cannot be covered by any finite sub-collection of $\{G_\alpha\}$ (otherwise, I could also be covered).

Now consider I_1 and subdivide it in a similar way, and continue this process indefinitely.

Thus we get a sequence $\{I_n\}$ of k-cells with the following properties :-

(i) $I \supset I_1 \supset I_2 \supset \cdots$

(ii) $\{G_\alpha\}$ does not have a finite subcover which covers I_n.

(iii) If $\bar{x} \in I_n$ & $\bar{y} \in I_n$, then $|\bar{x} - \bar{y}| \leq 2^{-n}\delta$

(This is because, consider I_1. We want to show $|\bar{x}-\bar{y}| \leq \delta/2$. Now $|\bar{x}-\bar{y}| = \sqrt{(y_1-x_1)^2 + (y_2-x_2)^2 + \cdots (y_k - x_k)^2}$

$$\leq \sqrt{\left(\frac{b_1-a_1}{2}\right)^2 + \left(\frac{b_2-a_2}{2}\right)^2 + \cdots \left(\frac{b_k-a_k}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(b_1-a_1)^2 + \cdots + (b_k-a_k)^2} = \frac{1}{2}\delta.$$

By (i) and Thm. 2.25, $\exists\ \bar{x}^* \ni \bar{x}^* \in I_n\ \forall n \in \mathbb{N}$.

Let $x^* \in G_\alpha$ for some α.

$\exists\ r > 0 \ni |\bar{y} - x^*| < r \Rightarrow y \in G_\alpha$.

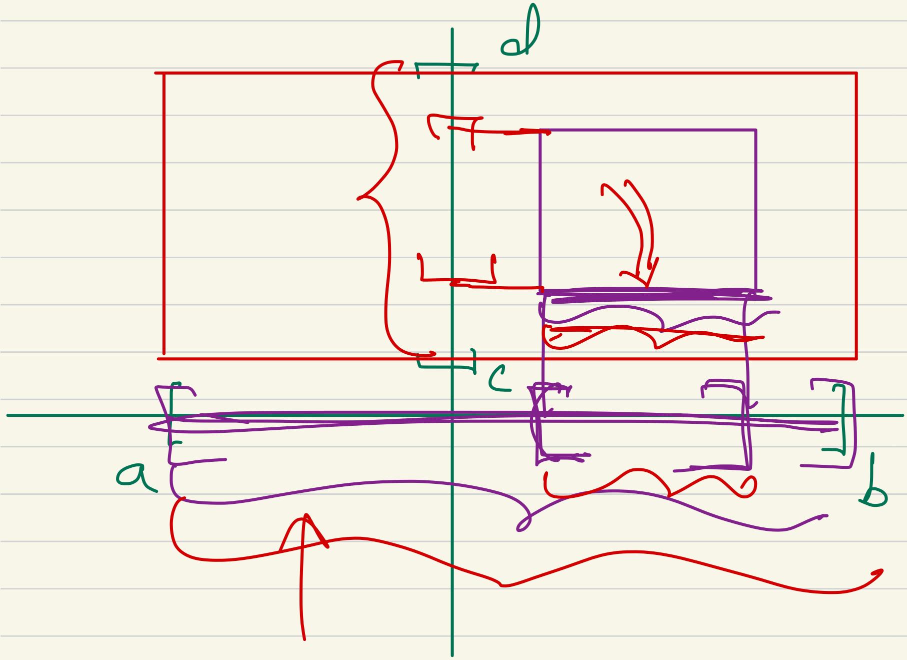
Choose n so large that $2^{-n}\delta < r$.

$\Rightarrow |\bar{x}^* - \bar{y}| < 2^{-n}\delta < r \Rightarrow \bar{y} \in G_\alpha$

Then $I_n \subset G_\alpha$ $\rightarrow\leftarrow$ since I_n did not have a finite subcover.

$\Rightarrow$ Every k-cell is compact.

$\boxtimes$



$$I_{1,j} \supset I_{2,j} \supset I_{3,j}$$