23/9/2020

MA 509 - REAL ANALYSIS LECTURE 18

Thm. 2.27 Let E be a set in $\mathbb{R}^k$. Then the following statements are equivalent:

a) E is closed and bounded
b) E is compact
} Heine-Borel theorem

c) Every infinite subset of E has a limit point in E.

Proof: a) $\Rightarrow$ b)

If E is bounded, then it can be enclosed within a k-cell. Moreover, if E is closed, since K is compact, Cor. 2.21 (of Lect. 15) implies that E must be compact.

b) $\Rightarrow$ c) Since E is compact, Thm. 2.23 implies that every infinite subset of E has a limit point in E.

c) $\Rightarrow$ a) (by contrapositivity)

Suppose c) holds but E is not bounded. Then E contains points $\bar{x}_n$ with

$$|\bar{x}_n| > n \quad , n \in \mathbb{N}.$$

The set S consisting of these points $\bar{x}_n$ is infinite and clearly has no limit point in $\mathbb{R}^k$ and hence none in E. $\Rightarrow\Leftarrow$

Thus c) implies that E is bounded.

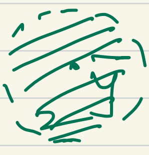
Suppose c) holds but E is not closed. Then $\exists\ \bar{x}_0 \in \mathbb{R}^k$ which is a limit point of E but not a point of E.

Then every nbhd of $\bar{x}_0$ intersects E in a point different from $\bar{x}_0$. In particular, take the nbhds $B(\bar{x}_0, \frac{1}{n})$, $n \in \mathbb{N}$. Then we get a sequence of points $\{\bar{x}_n\}_{n=1}^{\infty} \ni$

$$|\bar{x}_n - \bar{x}_0| < 1/n.$$

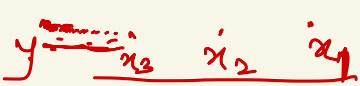
Let S be the set of all such points $\bar{x}_n$.
Then S must be infinite, for otherwise, $|\bar{x}_n - \bar{x}_0|$ would have the constant positive value for infinitely many n which will contradict the fact that $\bar{x}_0$ is a limit point of E.

Now S is an infinite subset of E having $\bar{x}_0$ as its limit point, and it cannot have any other limit point because if $\bar{y} \in \mathbb{R}^k$, $\bar{y} \neq \bar{x}_0$ is a limit point of S in $\mathbb{R}^k$, then



$$\begin{aligned}
|\bar{x}_n - \bar{y}| &= |(\bar{x}_n - \bar{x}_0) + (\bar{x}_0 - \bar{y})| \\
&= |(\bar{x}_0 - \bar{y}) - (\bar{x}_0 - \bar{x}_n)| \\
&\geq |\bar{x}_0 - \bar{y}| - |\bar{x}_n - \bar{x}_0| \quad \text{(Reverse triangle ineq.)} \\
&\geq |\bar{x}_0 - \bar{y}| - \frac{1}{n} \\
&\geq \frac{1}{2}|\bar{x}_0 - \bar{y}|.
\end{aligned}$$

Since $\frac{1}{2}|\bar{x}_0 - \bar{y}| \geq \frac{1}{n} \iff \underbrace{n|\bar{x}_0 - \bar{y}| \geq 2}_{\text{Follows from Archimedean property}}$



for all but infinitely many n so that $\bar{y}$ cannot be a limit point of S.

$\Rightarrow$ S has no limit point in E $\longrightarrow\!\!\longleftarrow$.
Hence E is closed if c) holds. ∎

Thm. 2.28 (Bolzano - Weierstrass Theorem)

Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Proof :- Let E be a bounded infinite subset of $\mathbb{R}^k$. Then E is enclosed within a k-cell I, which is itself a subset of $\mathbb{R}^k$.
Since I is compact and E is infinite, Thm. 2.26 implies that E has a limit pt. in I, and hence in $\mathbb{R}^k$. ∎