28/9/2020

MA 509 - REAL ANALYSIS - LECTURE 19

PERFECT SETS

Thm. 2.29 Let P be a non-empty perfect set in $\mathbb{R}^k$. Then P is uncountable.

Proof:- By defn., P is closed and every point of P is a limit point of P. Also, we are given that $P \neq \phi$.

Claim: P is uncountable.

First of all, since P has limit points, P has to be infinite.

By contradiction, suppose P is countable. Let $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots$ are elements of P.

We construct the following sequence of nbhds.

- V_1 is any nbhd of $\bar{x}_1$.
  Suppose $V_1 = \{\bar{y} \in \mathbb{R}^k : |\bar{y} - \bar{x}_1| < r\}$, and so $\bar{V}_1 = \{\bar{y} \in \mathbb{R}^k : |\bar{y} - \bar{x}_1| \leq r\}$.
- We inductively construct the other nbhds, i.e. suppose V_n has been constructed so that $V_n \cap P \neq \phi$.

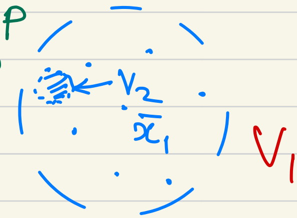
$\bar{V}_2 \subset V_1$, $\bar{x}_1 \notin \bar{V}_2$, $V_2 \cap P \neq \phi$

Since every point of P is a limit point of P, $\exists$ nbhd V_{n+1} s.t.

(i) $\overline{V_{n+1}} \subset V_n$ $\Rightarrow$ $\underbrace{\overline{V_{n+1}} \cap P}_{= K_{n+1}} \subset V_n \cap P \subset \underbrace{\overline{V_n} \cap P}_{= K_n}$

(ii) $\bar{x}_n \notin \overline{V_{n+1}}$

(iii) $V_{n+1} \cap P \neq \phi$.



Let $K_n = \overline{V_n} \cap P$.

Now $\overline{V_n}$ is closed and bounded, hence ~~so~~ in K_n, and thus K_n is compact.

Also $\bar{x}_n \notin \overline{V_{n+1}} \Rightarrow \bar{x}_n \notin K_{n+1}$.

Hence no point of P lies in $\bigcap_{n=1}^{\infty} K_n$.

But $K_n \subset P$ $\forall n \in \mathbb{N}$. Hence $\bigcap_{n=1}^{\infty} K_n \subset P$

$$\Rightarrow \bigcap_{n=1}^{\infty} K_n = \phi.$$

$V_n \cap P \neq \phi \Rightarrow \overline{V_n} \cap P \neq \phi$

But each K_n is non-empty by (iii).

Moreover $K_n \supset K_{n+1}$ $\forall n \in \mathbb{N}$ by (i).

This contradicts Cor. 2.22 & thus establishes that P is uncountable. $\blacksquare$

<u>Cor. 2.30</u> An interval $[a, b]$ $(a<b)$ is uncountable. In particular, $\mathbb{R}$ is uncountable

Proof: Use Thm. 2.29 with $P = [a, b]$

Problem 4 from Tutorial 5

* Does there exist a non-empty perfect set in $\mathbb{R}$ that contains no rational numbers?

Ans. Yes!

An example due to Julia Head, Eric Murphy & Brenton Calloway

Let $\{r_i\}_{i=1}^{\infty}$ be the sequence of rationals

Pick $\varepsilon_1 = k_1\sqrt{2}$, where $k_1 \in \mathbb{Q}^+$
$\Rightarrow \varepsilon_1 \in \mathbb{R}\setminus\mathbb{Q}$.

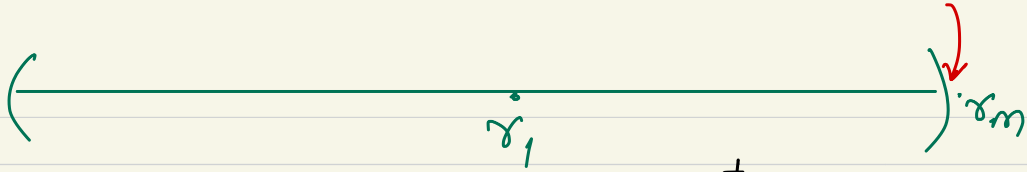
Consider $I_1 = (r_1 - \varepsilon_1, r_1 + \varepsilon_1)$

I_1 contains infinitely many rationals r_i.
But there are infinitely many $r_i \notin I_1$.

Pick the smallest such index $i \ni r_i \notin I_1$, say m. (Thus $r_i \notin I_1$ if $i \geq m$).

Let $\varepsilon_m = k_m\sqrt{2}$, where $k_m \in \mathbb{Q}^+$ and $\varepsilon_m <$ distance of r_m from I_1

Having constructed $I_1, I_2, \ldots, I_{n-1}$, let r_m be the rational with the smallest index m s.t. $r_n \notin I_1 \cup I_2 \cup I_3 \cup \cdots \cup I_{n-1}$



and let $\varepsilon_n = k_n\sqrt{2}$, $k_n \in \mathbb{Q}^+$ and s.t.

$I_n \cap \overline{I_1 \cup I_2 \cup \cdots \cdots \cup I_{n-1}} = \phi$.

First, $\mathbb{Q} \subset \bigcup_{n=1}^{\infty} I_n$ ——— (*)

$\bigcup_{n=1}^{\infty} I_n$ is open. ——— (1)

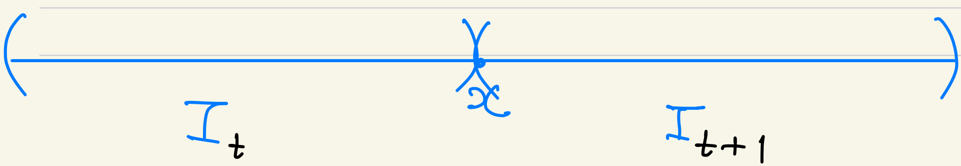
Let $P = \mathbb{R} \setminus \left(\bigcup_{n=1}^{\infty} I_n\right)$.

- P is closed (from (1))
- P contains no rationals (from (*))
- Claim: Every point of P is a limit point of P.

Proof: (by contradiction)
Suppose $\exists\, x \in P$ but $x \notin P'$.
$\Rightarrow \exists\, r > 0 \ni N_r(x) \cap P = \{x\}$.

$\Rightarrow N_r(x) \setminus \{x\} \subset \bigcup_{n=1}^{\infty} I_n$

$\Rightarrow$ x must be the open end point of 2 consecutive intervals



$\Rightarrow \exists\ i, j \in \mathbb{N}$ s.t.

$$x = r_i - \varepsilon_i = r_j + \varepsilon_j$$

$$\Rightarrow \underset{\in\,\mathbb{Q}}{r_i - r_j} = \varepsilon_i + \varepsilon_j = \underset{\in\,\mathbb{R}\setminus\mathbb{Q}}{\sqrt{2}(k_i + k_j)}$$

$\Rightarrow$ contradiction

Hence $x \in P'$.

$\Rightarrow$ P is perfect.