

Chapter 1 - The real and complex number systems

- The rational number system (consisting of the numbers of the form m/n , where $m, n \in \mathbb{Z}$ and $n \neq 0$) is inadequate as a field & $\underbrace{\mathbb{Z}}_{\text{(set of integers)}}$ also as an ordered set.

* For example, there is no rational p such that $p^2 = 2$.

Proof: Suppose p is a rational number m/n such that $p^2 = 2$. Then we could write $p = m/n$, $m, n \in \mathbb{Z}$, m & n are not both even. (For example, if we have $\frac{6}{14}$, it can be equivalently written as $\frac{27}{63}$.) Suppose this is done. Then $p^2 = 2 \Rightarrow m^2 = 2n^2$. So m^2 must be even, which implies that m must be even ($\because m$ odd $\Rightarrow m^2$ odd).

$\Rightarrow 4 \mid m^2 \Rightarrow 2 \mid n^2 \Rightarrow n^2$ is even $\Rightarrow n$ is even.

This contradicts our choice of m and n .

$\Rightarrow \nexists p \in \mathbb{Q} \ni p^2 = 2$.



* Let $A = \{p \in \mathbb{Q}^+ : p^2 < 2\}$ & $B = \{p \in \mathbb{Q}^+ : p^2 > 2\}$. Then A has no largest element and B has no smallest element.

Proof: For every $p \in A$, we find q in $A \ni p < q$, and for every p in B , we find a rational q in $B \ni q < p$. To show this, we associate with each rational $p > 0$, the number $q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2} \in \mathbb{Q}$. — (1)

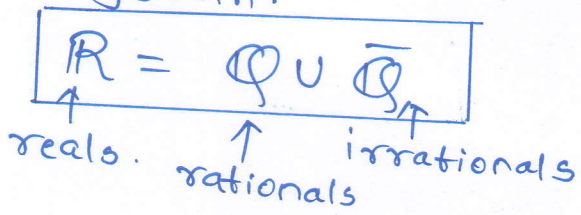
Then $q^2 - 2 = \frac{2(p^2 - 2)}{(p+2)^2}$ (2)

If $p \in A$, then $p^2 - 2 < 0$ (by defn.). From (1), we get $q > p$. But then (2) implies $q^2 < 2$. Hence $q \in A$.

Also, if $p \in B$, then $p^2 - 2 > 0$. So (1) shows $0 < q < p$, and thus from (2), $q^2 > 2$. Hence $q \in B$.

Implications:

- The rational number system has gaps inspite of the fact that between any two rationals, there is another rational: $r < \frac{r+s}{2} < s$.
- These gaps are filled by the irrational numbers.
- The rationals along with irrationals form the real number system.



ORDERED SETS

Let S be a set. An order on S is a relation, denoted by $<$, with the following properties:

(i) If $x \in S$ and $y \in S$, then one and only one of the following statements is true:

$x < y, x = y, x > y$

(ii) If $x, y, z \in S$ and $x < y, y < z$, then $x < z$.

• An ordered set is a set S on which an order is defined. (\mathbb{Q} is an ordered set if $r < s \Rightarrow s-r$ is a positive rational number.)