MA 509 – Real Analysis (Lecture 1)

Chapter 1 – The real and complex number systems

- The rational number system (consisting of the numbers of the form m/n, where $m, n \in \mathbb{Z}$ (set of integers) and $n \neq 0$) is inadequate as a field & also as an ordered set.

* For example, there is no rational p such that $p^2 = 2$.

Proof: Suppose p is a rational number m/n such that $p^2 = 2$. Then we could write $p = m/n$, $m, n \in \mathbb{Z}$, m & n are not both even. (For example, if we have $\frac{6}{14}$, it can be equivalently written as $27/63$.) Suppose this is done. Then $p^2 = 2 \Rightarrow m^2 = 2n^2$. So m^2 must be even, which implies that m must be even ($\because$ m odd $\Rightarrow m^2$ odd).
$\Rightarrow 4 \mid m^2 \Rightarrow 2 \mid n^2 \Rightarrow n^2$ is even $\Rightarrow n$ is even.
This contradicts our choice of m and n.
$\Rightarrow \nexists\, p \in \mathbb{Q} \ni p^2 = 2$.

∎

* Let $A = \{ p \in \mathbb{Q}^+ : p^2 < 2 \}$ & $B = \{ p \in \mathbb{Q}^+ : p^2 > 2 \}$.
Then A has no largest element and B has no smallest element.

Proof: For every $p \in A$, we find q in $A \ni p < q$, and for every p in B, we find a rational q in $B \ni q < p$.
To show this, we associate with each rational $p > 0$, the number

$$q = p - \frac{p^2 - 2}{p+2} = \frac{2p+2}{p+2} \in \mathbb{Q}. \quad \text{——} \quad (1)$$

Then $q^2 - 2 = \dfrac{2(p^2-2)}{(p+2)^2}$ ——— (2)

If $p \in A$, then $p^2 - 2 < 0$ (by defn.). From (1), we get $q > p$. But then (2) implies $q^2 < 2$. Hence $q \in A$.

Also, if $p \in B$, then $p^2 - 2 > 0$. So (1) shows $0 < q < p$, and thus from (2), $q^2 > 2$. Hence $q \in B$. ▨

Implications:

- The rational number system has gaps inspite of the fact that between any two rationals, there is another rational: $r < \frac{r+s}{2} < s$.
- These gaps are filled by the irrational numbers.
- The rationals along with irrationals form the real number system.

$$\boxed{\underset{\text{reals}}{\mathbb{R}} = \underset{\text{rationals}}{\mathbb{Q}} \cup \underset{\text{irrationals}}{\overline{\mathbb{Q}}}}$$

ORDERED SETS

Let S be a set. An order on S is a relation, denoted by $<$, with the following properties:

(i) If $x \in S$ and $y \in S$, then one and only one of the following statements is true:

$$x < y, \quad x = y, \quad x > y$$

(ii) If $x, y, z \in S$ and $x < y$, $y < z$, then $x < z$.

- An ordered set is a set S on which an order is defined. ($\mathbb{Q}$ is an ordered set if $r < s \Rightarrow s - r$ is a positive rational number)