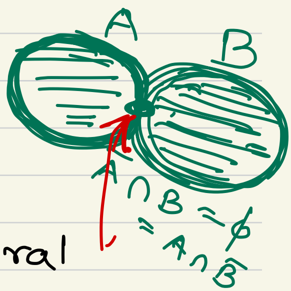
29/9/2020

MA 509 - REAL ANALYSIS - LECT. 21

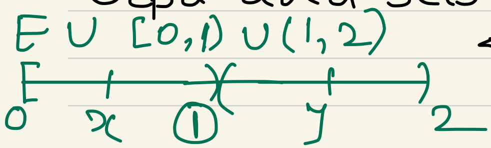
CONNECTED SETS

Defn. Two subsets A and B of a metric space X are said to be separated if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty.

A set $E \subset X$ is connected if E is NOT a union of 2 non-empty separated sets.



Separated sets $\longrightarrow$ disjoint



$\longleftarrow$ not true in general

Eg. $[0,1]$ and $(1,2)$ are disjoint but not separated, since $[0,1] \cap \overline{(1,2)} = \{1\} \neq \phi$.

But $(0,1)$ and $(1,2)$ are separated.

Q: How do connected subsets of real line look?

Thm. 2.30 A subset E of $\mathbb{R}$ is connected iff it has the following property:

If $x \in E, y \in E$ and $x < z < y$, then $z \in E$.

"$\Longrightarrow$" (by contrapositivity)

Proof: Suppose $\exists\, x, y \in E$ and $z \in (x,y)$ s.t. $z \notin E$, then

$E = A_z \cup B_z$, where $A_z = E \cap (-\infty, z)$
& $B_z = E \cap (z, \infty)$.

Now $x \in A_z$ and $y \in B_z$, so $A_z \neq \phi$, $B_z \neq \phi$.

Note that $\overline{(-\infty, z)} \cap (z, \infty) = \phi$ &
$(-\infty, z) \cap \overline{(z, \infty)} = \phi$

& since $A_z \in (-\infty, z)$ & $B_z \in (z, \infty)$, they are separated. Hence E is not connected.

"$\Longleftarrow$" (by contrapositivity)

Suppose E is not connected. Then $\exists$ non-empty separated sets A & B $\ni$ $E = A \cup B$.

Let $x \in A$ & $y \in B$ & w.l.g. let $x < y$.

Define $z = \sup(A \cap [x,y])$.

Then $\underbrace{z \in \overline{A \cap [x,y]}}_{\text{Thm. 2.16 (Lect. 14)}} \subset \bar{A}$.

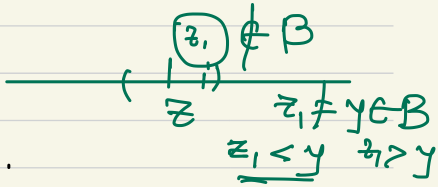
Since $\bar{A} \cap B = \phi$, $z \notin B$. — (1)

$\Longrightarrow$ $x \leq z < y$

Case 1 : $z \notin A$

Then along with (1), we have $z \notin A \cup B = E$. $\rightarrow\leftarrow$ to the given hypothesis.

Case 2 : $z \in A$



Then $A \cap \overline{B} = \phi \Rightarrow z \notin \overline{B}$

Thus $z \notin B'$, in particular.

$\Rightarrow \exists$ a nbhd of z which has empty intersection with B, i.e., $\exists\, z_1 \ni z < z_1 < y$ and $z_1 \notin B$. — (2)

$\Rightarrow x \leq z < z_1 < y$ — (3)

Now $z_1 \notin A$, for, otherwise, it contradicts the fact that $z = \sup(A \cap [x, y])$

$\Rightarrow$ From (2) & (3), $\exists\, z_1 \ni x < z_1 < y$ & s.t. $z_1 \notin E$. $\rightarrow\leftarrow$ to the given hypothesis.

$\blacksquare$

Chapter 3 – Numerical sequences & series

Convergent sequences

Def. A sequence $\{p_n\}$ in a metric space X is said to converge if $\exists\, p \in X \ni$ for every $\varepsilon > 0$, $\exists\, N \in \mathbb{N} \ni n \geq N$ implies $d(p_n, p) < \varepsilon$.

Then we say $\{p_n\}$ converges to p, or p is a limit point of $\{p_n\}$.

We say $p_n \to p$ or $\lim_{n \to \infty} p_n = p$.

If $\{p_n\}$ does not converge, we say it diverges.

Remark: Not only does it matter whether $\{p_n\}$ converges or not, but also where, i.e., in which metric space it converges matters too.

e.g. ① $\{\frac{1}{n}\} \to 0$ in $\mathbb{R}$ but not in $\mathbb{R}^+$

② $\left\{ \frac{1}{\sqrt{n}} : n \in \mathbb{N}, \nexists\, m \in \mathbb{N} \ni m^2 = n \right\}$

$\longrightarrow 0$ in $\mathbb{R}$ but not in $\mathbb{R} \setminus \mathbb{Q}$.

- Range of $\{p_n\}$: All points p_n, $n \in \mathbb{N}$.
- $\{p_n\}$ is said to be bounded if its range is bounded.

Let the metric space X be $\mathbb{R}^2$.

	Range	Bounded?	convergent?
(a) $s_n = \frac{1}{n}$	infinite	bdd	Yes
(b) $s_n = n^2$	infinite	unbdd	No
(c) $s_n = 1 + \frac{(-1)^n}{n}$	infinite	bdd	Yes
(d) $s_n = i^n$	finite	bdd	No
(e) $s_n = 1 \ \forall n \in \mathbb{N}$	finite	bdd	Yes