MA 509 – REAL ANALYSIS — LECTURE 26 19/10/2020

Examples contd.

③ For a real-valued sequence $\{s_n\}$, $\lim_{n\to\infty} s_n = s$ iff

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s.$$

(This will be one of the problems in the upcoming tutorial on Friday.)

Thm. 3.9 If $s_n \leq t_n$ for $n \geq N$, where N is fixed,

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$$

$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n .$$

Some special sequences

Thm. 3.10

ⓐ If $p>0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

Given $\varepsilon>0$,

Choose $N \in \mathbb{N} \ni N > (1/\varepsilon)^{1/p} \Rightarrow N^p > 1/\varepsilon \Rightarrow \frac{1}{N^p} < \varepsilon$

Now for $n \geq N$, $\frac{1}{n} \leq \frac{1}{N} \Rightarrow \frac{1}{n^p} \leq \frac{1}{N^p} < \varepsilon$

$\Rightarrow \left|\frac{1}{n^p} - 0\right| < \varepsilon$ for $n \geq N \Rightarrow \frac{1}{n^p} \to 0$.

ⓑ If $p>0$, then $\lim_{n\to\infty} p^{1/n} = 1$.

Proof: Case ① : $p>1$

Let $x_n = p^{1/n} - 1$. Then $x_n > 0$.
Now binomial theorem gives

$$1 + n x_n \leq (1 + x_n)^n = p$$

$$\Rightarrow \quad 0 < x_n \leq \frac{p-1}{n}$$

$$\Rightarrow \quad x_n \to 0$$

If $0 \leq x_n \leq s_n$ for $n \geq N$, where N is fixed, and if $s_n \to 0$, then $x_n \to 0$.

Case 2: $p = 1$: trivial

Case 3: $0 < p < 1$. Then $1/p > 1$.
So by case 1, $\lim_{n \to \infty} \left(\frac{1}{p}\right)^{\frac{1}{n}} = 1$

$$\Rightarrow \lim_{n \to \infty} p^{1/n} = 1.$$

(c) $\lim_{n \to \infty} n^{1/n} = 1$

Proof: Let $x_n = n^{1/n} - 1$. Then $x_n \geq 0$.
Using binomial thm., we have

$$n = (1 + x_n)^n \geq \frac{n(n-1)}{2} x_n^2$$

$$\Rightarrow \quad 0 \leq x_n \leq \sqrt{\frac{2}{n-1}} \quad \text{for } n \geq 2.$$

Since $\lim_{n \to \infty} \sqrt{\frac{2}{n-1}} = 0$, we get $\lim_{n \to \infty} x_n = 0$

$$\Rightarrow \lim_{n \to \infty} n^{1/n} = 1.$$

(d) If $p>0$ & $\alpha \in \mathbb{R}$, $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

Proof: Let $k \in \mathbb{Z} \ni k > \alpha$, $k > 0$. Note that

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k$$

Now if we choose $n \ni n > 2k$, then

$$n-k+1 > \frac{n}{2} \Rightarrow n(n-1)\cdots(n-k+1) > \underbrace{\frac{n}{2}\cdot\frac{n}{2}\cdots\frac{n}{2}}_{k \text{ times}}$$

$\frac{n}{2} > k-1$

$$= \left(\frac{n}{2}\right)^k$$

$$\Rightarrow (1+p)^n > \frac{n^k p^k}{2^k k!} \quad \text{so that} \quad \frac{1}{(1+p)^n} < \frac{2^k k!}{n^k p^k}$$

$$\Rightarrow 0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} \cdot n^{\alpha-k} \quad (\text{for } n > 2k)$$

$\alpha < k \Rightarrow n^{\alpha-k} \to 0$ as $n \to \infty$ by part (a).

$$\Rightarrow \lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0.$$

(e) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

Proof: Let $\alpha = 0$ in (d), so that

$$\lim_{n\to\infty} \frac{1}{(1+p)^n} = 0 \longrightarrow (*)$$

Case 1: $x = 0$. trivial

$\frac{1}{x} > 1$

Case 2: $0 < x < 1$. Let $p = \frac{1}{x} - 1$. Then $p > 0$.

From (*), $\lim_{n\to\infty} x^n = 0$ since $x = \frac{1}{1+p}$.

Case 3: $-1 < x < 0$ Then $-1 - \frac{1}{x} > 0$.

$1+p = \frac{-1}{x} \Rightarrow -x = \frac{1}{1+p}$

Let $p = -1 - \frac{1}{x}$. Also $-x = \frac{1}{1+p}$

By (d), $\lim_{x\to\infty} (-x)^n = 0$.

$$\Rightarrow \lim_{x\to\infty} x^n = 0.$$

SERIES Given a sequence $\{a_n\}$, let

$\sum_{n=p}^{q} a_n$ denote the sum $a_p + a_{p+1} + \cdots + a_q$,

where $p \leq q$.

Then associated to the sequence $\{a_n\}$ is the sequence $S_n = \sum_{k=1}^{n} a_k$.

$S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$

Notation: The series $\sum_{k=1}^{\infty} a_k$ is called the infinite series and is used to denote $a_1 + a_2 + a_3 + \cdots$

The numbers S_n are called the partial sums of the series.

If $\{s_n\}$ converges to s, we say that the (infinite) series converges & write

$$\sum_{k=1}^{\infty} a_k = s.$$

($\sum_{k=1}^{\infty} a_k$ is to be interpreted as $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$)

- If $\{s_n\}$ diverges, we say the (infinite) series diverges.
- Note that $a_1 = s_1$ & for $n > 1$, $a_n = s_n - s_{n-1}$.

CAUCHY CRITERION FOR SERIES

Henceforth, $a_n \in \mathbb{C}$, unless specified otherwise.

Thm. 3.11 $\sum a_n$ converges iff for every $\varepsilon > 0$, $\exists N \in \mathbb{N} \ni$ $\left| \sum_{k=n}^{m} a_k \right| \leq \varepsilon$ if $m \geq n \geq N$

Note that $\sum a_n$ conv. $\Rightarrow \{s_n\}$ converges $\Rightarrow \{s_n\}$ is Cauchy

Given $\varepsilon > 0$, $\exists N \in \mathbb{N} \ni \forall m \geq n \geq N$, $|s_m - s_{n-1}| \leq \varepsilon$

$$\Rightarrow \left| \sum_{k=n}^{m} a_k \right| \leq \varepsilon$$

Thm. 3.12 If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Proof: Take $m = n$ in Thm. 3.11. Then

$$|a_n| \leq \varepsilon \quad \text{for } n \geq N, \text{ i.e.,}$$

$$|a_n - 0| \leq \varepsilon \quad \text{for } n \geq N$$

By defn., this implies $a_n \to 0$ as $n \to \infty$.

WARNING The condition $a_n \to 0$ is NOT sufficient to ensure convergence of $\sum a_n$.

series of non-negative terms $\Rightarrow$ $\{S_n\}$ is monot. incr.

Thm. 3.13 A series of non-negative terms converges iff its partial sums form a bounded sequence.