MA 509 – REAL ANALYSIS – LECTURE 28

26/10/2020

Thm. 3.19 $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

Proof: Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$, $t_n = \left(1+\frac{1}{n}\right)^n$

By the binomial theorem,

$$t_n = 1 + n\cdot\frac{1}{n} + \frac{1}{2!}\frac{n(n-1)}{n^2} + \frac{1}{3!}\frac{n(n-1)(n-2)}{n^3} + \dots + \frac{1}{n!}\cdot\frac{n(n-1)\cdots 2\cdot 1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

Thus $t_n \le s_n$ & hence

$$\limsup_{n\to\infty} t_n \le \limsup_{n\to\infty} s_n = e \quad \text{—— (1)}$$

$$\left(\because s_n = \sum_{k=0}^{n}\frac{1}{k!} \to e \text{ as } n\to\infty\right)$$

Next, if $n \geq m$,

$$t_n \geq 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \dots + \frac{1}{m!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{(m-1)}{n}\right)$$

Fix m and let $n\to\infty$. Then

$$\liminf_{n\to\infty} t_n \geq 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{m!} = s_m.$$

Now let $m \to \infty$,

$$e \leq \liminf_{n\to\infty} t_n. \quad \text{—— (2)}$$

$\Rightarrow$

$$e \leq \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n \leq e$$

so that

$$\lim_{n\to\infty} t_n = e$$

$\boxtimes$

<u>Remark</u>: $e - s_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots$

$$< \frac{1}{(n+1)!}\left\{1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right\} = \frac{1}{(n+1)!} \frac{1}{\left(1 - \frac{1}{n+1}\right)}$$

$$= \frac{1}{n!\, n}$$

$$\Rightarrow \quad 0 < e - s_n < \frac{1}{n!\,n}.$$

<u>Thm. 3.20</u> e is irrational

<u>Proof</u>: Suppose e is rational, then $e = p/q$ where $p, q \in \mathbb{N}$. Then from the above remark,

$$0 < q!\,(e - s_q) < \frac{1}{q} \quad \text{—— } (*)$$

Now $e = \frac{p}{q}$ implies $q!\,e \in \mathbb{N}$

Since $q!\, s_q = q!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!}\right) \in \mathbb{N}$,

we find that $q!\,(e - s_q) \in \mathbb{N}$.

Since $q \geq 1$, $\frac{1}{q} \leq 1$, and so (*) implies

existence of an integer between 0 & 1.

$\rightarrow\leftarrow$

Hence e is irrational. $\blacksquare$

ROOT TEST

Thm. 3.21 Given $\sum a_n$, put $\alpha = \limsup\limits_{n\to\infty} |a_n|^{1/n}$.
Then

(a) if $\alpha < 1$, $\sum a_n$ converges
(b) if $\alpha > 1$, $\sum a_n$ diverges
(c) if $\alpha = 1$, the test is inconclusive.

Proof: Let $\alpha < 1$. Choose $\beta \ni \alpha < \beta < 1$. Since $\beta > \alpha$, $\exists\, N \in \mathbb{N} \ni \forall n \geq N$,

$$|a_n|^{1/n} < \beta$$

$\Rightarrow |a_n| \leq \beta^n$ for all $n \geq N$.

Now $0 < \beta < 1$, so $\sum \beta^n$ converges.

Hence $\sum a_n$ converges by comparison test.

If $\alpha > 1$, $\exists$ seq. $\{n_k\}$ such that

$$|a_{n_k}|^{1/n_k} \to \alpha \qquad |a_{n_k}| > 1^{n_k} = 1$$

Thus $|a_n| > 1$ for ∞'ly many values of n.
Hence $a_n \not\to 0$ as $n \to \infty$.
Hence $\sum a_n$ diverges.

If $\alpha = 1$, the test is inconclusive since take for example, $\sum \frac{1}{n}$ & $\sum \frac{1}{n^2}$.

$\alpha = 1$ in each case. But $\sum \frac{1}{n}$ diverges whereas $\sum \frac{1}{n^2}$ converges.

$$\alpha = \limsup_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = \limsup_{n\to\infty} \frac{1}{n^{1/n}} = 1$$

$$\limsup_{n\to\infty} \frac{1}{n^{2/n}} = 1$$

RATIO TEST

Thm. 3.22 The series $\sum a_n$

ⓐ converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,

ⓑ diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for $n \geq n_0$, where n_0 is some fixed integer.

Proof: If ⓐ holds, we can find $\beta < 1$ & $N \in \mathbb{N} \ni$ $\forall n \geq N$,

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta.$$

In particular, $|a_{N+1}| < \beta |a_N|$

$|a_{N+2}| < \beta |a_{N+1}| < \beta^2 |a_N|$

$\vdots$

$|a_{N+p}| < \beta^{p} |a_N|$ $\quad n-N$

$\Rightarrow |a_n| < |a_N| \beta^{n-N}$ for $n \geq N$.

$$\sum_{n=N}^{\infty} \beta^{n-N}$$

$n = m+N$

$$\sum_{m=0}^{\infty} \beta^{m}$$

Since $\sum \beta^{n-N}$ converges ($\beta < 1$), we see by comp. test that $\sum a_n$ converges.

If $|a_{n+1}| \geq |a_n|$ for $n \geq n_0$, then $a_n \not\to 0$ as $n \to \infty$. Hence $\sum a_n$ diverges. $|a_{n_0+1}| \geq |a_{n_0}|$

$|a_{n_0+2}| \geq$

$\leq |a_{n_0+3}|$ ...

<u>Examples</u> (a) Consider

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \boxed{\left(\frac{2}{3}\right)^n} = 0.$$

$\boxed{\frac{2}{3}}, \frac{3}{4}, \boxed{\frac{4}{9}}, \frac{9}{8}, \boxed{\frac{8}{27}}, \ldots$

$$\liminf_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \sqrt[2n]{\frac{1}{3^n}} = \frac{1}{\sqrt{3}}.$$

$a_1^{1/1} = \frac{1}{2}$ $\quad a_2^{1/2} = \frac{1}{\sqrt{3}}$ $\quad a_3^{1/3} = \frac{1}{2^{2/3}}$

$a_4^{1/4} = \left(\frac{1}{3^2}\right)^{\frac{1}{4}} = \frac{1}{\sqrt{3}}$, $\quad a_5^{1/5} = \frac{1}{2^{3/5}}$

$$a_6^{1/6} = \frac{1}{\sqrt{3}}, \quad a_7^{1/7} = \frac{1}{2^{4/7}}$$

$$\frac{1}{2^1}, \frac{1}{2^{2/3}}, \frac{1}{2^{3/5}}, \frac{1}{2^{4/7}} \cdots$$

$$\lim_{n\to\infty} \frac{1}{2^{\frac{n}{2n-1}}} = \lim_{n\to\infty} \frac{1}{2^{\frac{1}{2-1/n}}} = \frac{1}{\sqrt{2}}.$$

$$\sqrt{2} < \sqrt{3} \quad \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}}$$

$$\frac{2}{3}, \frac{3}{4}, \frac{4}{9}, \frac{9}{8}, \frac{8}{27}, \frac{27}{16} \cdots$$

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = +\infty$$

$$\limsup_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{1}{(2^n)^{\frac{1}{2n-1}}} = \boxed{\frac{1}{\sqrt{2}}} < 1$$

$\Rightarrow$ So the root test says $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$ is convergent.