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MA 509 - REAL ANALYSIS - LECTURE 29

Thm. 3.23 For any sequence $\{c_n\}$ of positive numbers,

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} c_n^{1/n}$$

$$\limsup_{n\to\infty} c_n^{1/n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

POWER SERIES

Defn. Given a sequence $\{c_n\}$ of complex numbers, the series $\sum_{n=0}^{\infty} c_n z^n$ is called a power series.

- Associated to every power series is a circle (of convergence) such that the above power series converges if z is in the interior of the circle, and diverges if z is in the exterior.

On the circle itself, the behavior is more varied.

Thm. 3.24 Given the power series $\sum c_n z^n$, put $\alpha = \limsup_{n\to\infty} |c_n|^{1/n}$ & $R = 1/\alpha$.

Then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$.

Proof:- Apply root test with $a_n = c_n z^n$ so that

$$\limsup_{n\to\infty} |a_n|^{1/n} = |z| \limsup_{n\to\infty} |c_n|^{1/n} = \frac{|z|}{R}. \quad \blacksquare$$

* R is called the radius of convergence (r.o.c.) of the power series.

Examples

① $\sum n^n z^n$ has $R = 0$.

① $c_n = n^n$; $\limsup_{n\to\infty} |c_n|^{1/n} = \limsup_{n\to\infty} n = +\infty \Rightarrow \alpha = \infty$.

$\Rightarrow R = 1/\alpha = 0$.

② $\sum \frac{z^n}{n!}$ has $R = +\infty$.

③ $\sum z^n$ has $R = 1$. The series diverges for $|z| = 1$ since $\{z^n\} \not\to 0$ as $n \to \infty$.

④ $\sum \frac{z^n}{n}$ has $R = 1$. For $z = 1$, it diverges. It converges for $|z| = 1$, $z \neq 1$.

⑤ $\sum \frac{z^n}{n^2}$ has $R = 1$. It converges for all z with $|z| = 1$, since $\left|\frac{z^n}{n^2}\right| = \frac{1}{n^2}$.

② $a_n = \frac{z^n}{n!}$ $\quad \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty} \left|\frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n}\right|$

$$= \limsup_{n\to\infty} \frac{|z|}{n+1} = |z| \limsup_{n\to\infty} \frac{1}{n+1}$$

$$= |z| \cdot 0$$

$$= 0 < 1 \quad \text{for any } z \in \mathbb{C}.$$

$\Rightarrow$ Ratio test implies $\sum z^n/n!$ conv. $\forall z \in \mathbb{C}$. $\Rightarrow R = \infty$.

③ $\sum z^n$. $c_n = 1 \ \forall n \in \mathbb{N} \cup \{0\}$

$$R = \frac{1}{\limsup_{n\to\infty} c_n^{1/n}} = \frac{1}{1} = 1.$$

④ $\sum \frac{z^n}{n}$ $\quad R = \frac{1}{\limsup_{n\to\infty} c_n^{1/n}} = \frac{1}{\limsup_{n\to\infty} \frac{1}{n^{1/n}}}$

$c_n = 1/n \ \forall n \in \mathbb{N}$

$$= \frac{1}{1} = 1.$$

⑤ Suppose $|z| = 1$.

$$\left|\sum_{n=1}^{\infty} \frac{z^n}{n^2}\right| \leq \sum_{n=1}^{\infty} \frac{|z|^n}{n^2} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{converges.}}$$

By comparison test, $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges.

SUMMATION BY PARTS

Thm. 3.25 Given 2 sequences $\{a_n\}$ & $\{b_n\}$, let $A_n = \sum_{k=0}^{n} a_k$, if $n \geq 0$ & let $A_{-1} = 0$.

Then, if $0 \le p \le q$,

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p .$$

Proof: See Rudin.

- The above 'partial summation formula' helps us in investigating the series of the form $\sum a_n b_n$, especially when $\{b_n\}$ is montonic.

<u>Applications of the partial summation formula</u>

<u>Thm.3.26</u> Suppose

(a) the partial sums A_n of $\sum a_n$ form a bounded sequence;

(b) $b_0 \geq b_1 \geq b_2 \geq \dots$;

(c) $\lim_{n\to\infty} b_n = 0$.

Then $\sum a_n b_n$ converges.

<u>Proof</u>: By (a), $\exists M > 0 \ni |A_n| \le M \ \forall n \in \mathbb{N}$.
By (c), given an $\varepsilon > 0$, $\exists N \in \mathbb{N} \ni b_N \le \frac{\varepsilon}{2M}$.

By the partial summation formula, for $N \le p \le q$,

$$\left| \sum_{n=p}^{q} a_n b_n \right| = \left| \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \right|$$

$$\le \sum_{n=p}^{q-1} |A_n| \, |b_n - b_{n+1}| + |A_q| b_q + |A_{p-1}| b_p$$

$$= \sum_{n=p}^{q-1} |A_n|(b_n - b_{n+1}) + |A_q|\, b_q + |A_{p-1}|\, b_p \quad (\because b_n \geq b_{n+1})$$

$$\leq M\left(\sum_{n=p}^{q-1}(b_n - b_{n+1}) + b_q + b_p\right)$$

$$= M\left(b_p - \cancel{b_{p+1}} + \cancel{b_{p+1}} - \cancel{b_{p+2}} + \cancel{b_{p+2}} - \cancel{b_{p+3}} + \cancel{\cdots} + b_{q-1} - \cancel{b_q} + \cancel{b_q} + b_p\right)$$

$$= 2Mb_p \leq 2Mb_N \leq 2M \cdot \frac{\varepsilon}{2M} = \varepsilon.$$

Hence $\sum a_n b_n$ converges by Cauchy criterion.

Thm. 3.27 (Alternating series test)

(a) $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$

(b) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ $(\forall m \in \mathbb{N})$

(c) $\lim_{n\to\infty} c_n = 0$.

Then $\sum c_n$ converges.

Proof: Let $a_n = (-1)^{n+1}$, $b_n = |c_n|$ in Thm. 3.26.

Note that $A_n = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} (-1)^{k+1} \leq 2$

Also by (c), $\lim_{n\to\infty} |c_n| = 0$. So by Thm. 3.26,

$\sum (-1)^{n+1}|c_n|$ converges $\sum_{n=1}^{\infty}(-1)^{n+1} c_n = c_1 - (-c_2) + c_3 - (-c_4) + \cdots = c_1 + c_2 + c_3 + \cdots$

Thm. 3.28 Suppose the radius of convergence of $\sum c_n z^n$ is 1 and suppose $c_0 \geq c_1 \geq c_2 \geq \cdots$, $\lim_{n\to\infty} c_n = 0$. Then $\sum c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly at $z = 1$.

Proof:- Let $a_n = z^n$ & $b_n = c_n$. The hypotheses of Thm. 3.26 are satisfied since

$$|A_n| = \left|\sum_{m=0}^{n} z^m\right| = \left|\frac{1 - z^{m+1}}{1 - z}\right| \leq \frac{2}{|1-z|},$$

if $|z| = 1$, $z \neq 1$.

Example

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$$\sum \frac{z^n}{n}$$ conv. for all $z \ni |z| \leq 1,\ z \neq 1$

$c_n = \frac{1}{n}$ Apply Thm. 3.28 to conclude