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MA 509 – REAL ANALYSIS – LECTURE 29

ABSOLUTE CONVERGENCE

$\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges. e.g. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is abs. conv. ser

Thm. 3.29 If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

(Absolute convergence $\Longrightarrow$ convergence)

Proof: Note that

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_m|.$$

Now use Cauchy criterion.

- There are series which converge without converging absolutely. Such series are called conditionally convergent series.

Example: $\sum \frac{(-1)^n}{n}$.

ADDITION AND MULTIPLICATION OF SERIES

Thm. 3.30 If $\sum a_n = A$, $\sum b_n = B$, then

$$\sum (a_n + b_n) = A + B$$

$$\sum c a_n = c \sum a_n \quad \text{for any fixed } c.$$

CAUCHY PRODUCT

Given $\sum a_n$ & $\sum b_n$, put

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \qquad (n=0,1,2,\ldots)$$

Then $\sum c_n$ is called the Cauchy product of 2 series since $\sum c_n = \sum a_n \sum b_n$.

MOTIVATION

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right)$$

$$= (a_0 + a_1 z + a_2 z^2 + \ldots)(b_0 + b_1 z + b_2 z^2 + \ldots)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \ldots$$

$$= c_0 + c_1 z + c_2 z^2 + \ldots$$

Now let $z=1$.

Example: If $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, $C_n = \sum_{k=0}^{n} c_k$ & $A_n \to A$, $B_n \to B$, then C_n does not necessarily converge to AB.

Consider $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$. It converges by the alternating series test.

$\sum \frac{(-1)^n}{\sqrt{n+1}}$ converges because $\left\{\frac{1}{\sqrt{n+1}}\right\} \longrightarrow 0$ as is a decr. seq.

Let $a_n = b_n = \dfrac{(-1)^n}{\sqrt{n+1}}$.

Then $c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} \dfrac{(-1)^k}{\sqrt{k+1}} \dfrac{(-1)^{n-k}}{\sqrt{n-k+1}}$

$$= (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}}.$$

Now $(n-k+1)(k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \leq \left(\frac{n}{2}+1\right)^2$.

Then $|c_n| \geq \sum_{k=0}^{n} \dfrac{1}{\left(\frac{n}{2}+1\right)} = \dfrac{2(n+1)}{(n+2)} \not\to 0$ as $n \to \infty$.

<u>Thm. 3.31</u> (<u>Mertens</u>) Suppose

(a) $\sum_{n=0}^{\infty} a_n$ converges absolutely;

(b) $\sum_{n=0}^{\infty} a_n = A$;

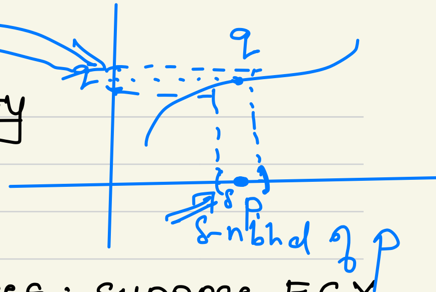
(c) $\sum_{n=0}^{\infty} b_n = B$;

(d) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ $(n = 0, 1, 2, \ldots)$

Then $\sum_{n=0}^{\infty} c_n = AB$.

<u>Thm. 3.32</u> If the series $\sum a_n$, $\sum b_n$ & $\sum c_n$ converge to A, B, C & $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then $C = AB$.

Chapter 4 – Continuity



Limits of functions

Let X and Y be metric spaces; suppose $E \subset X$, $f : E \to Y$ and p is a limit point of E. Then we say $f(x) \to q$ as $x \to p$, or in other words,

$$\lim_{x \to p} f(x) = q$$

if $\exists\, q \in Y$ with ~~the~~ following property:

For every $\varepsilon > 0$, $\exists\, \delta > 0 \ni d_Y(f(x), q) < \varepsilon$ for all $x \in E$ for which $0 < d_X(x, p) < \delta$.

- p may or may not be a point of E.
- Even if $p \in E$, we may have $f(p) \neq \lim_{x \to p} f(x)$.

Thm. 4.1 Let X, Y, E, f and p be defined as before. Then $\lim_{x \to p} f(x) = q$ iff $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in $E \ni p_n \neq p$, $\lim_{n \to \infty} p_n = p$.

$\Rightarrow$

Proof:- Suppose $\lim_{x \to p} f(x) = q$. Choose $\{p_n\}$ in E $\ni p_n \to p$. Let $\varepsilon > 0$ be given. Then $\exists\, \delta > 0 \ni d_Y(f(x), p) < \varepsilon$ whenever $x \in E$ & $0 < d_X(x, p) < \delta$.

Also, $\exists N \in \mathbb{N} \ni \forall n > N$, $0 < d_X(p_n, p) < \delta$.

Thus for all $n > N$, $d_Y(f(p_n), q) < \varepsilon$

$\Rightarrow \lim_{n\to\infty} f(p_n) = q$.

Given $\varepsilon > 0, \exists \delta > 0 \ni d_X(x, p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon$

"$\Leftarrow$" Conversely, suppose $\lim_{x\to p} f(x) \neq q$. This implies $\exists \varepsilon > 0 \ni \forall \delta > 0$, $\exists x \in E$ (depending on δ) s.t. $d_Y(f(x), q) \geq \varepsilon$ but $0 < d_X(x, p) < \delta$.

Take $\delta_n = 1/n$, $n \in \mathbb{N}$, in particular. Then we can find a seq. $\{p_n\} \ni p_n \to p$ but $d_Y(f(p_n), q) \geq \varepsilon$.

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Cor. 4.2 If f has a limit at p, this limit is unique.

In a metric space, if $\underbrace{p_n \to p}$ & $p_n \to p'$, then $p = p'$.

$\lim_{x\to p} f(x) = q$, say. (by the hypothesis)

Spse $\lim_{x\to p} f(x) = q'$, where $q' \neq q$, then $\lim_{n\to\infty} f(p_n) = q'$

But $\lim_{n\to\infty} f(p_n) = q$. Hence $q = q'$.

Thm. 4.3 Suppose $E \subset X$, a metric space, p is a limit point of E, f and g are complex functions on E, and $\lim_{x\to p} f(x) = A$, $\lim_{x\to p} g(x) = B$. Then

(a) $\lim_{x\to p} (f+g)(x) = A + B$;

(b) $\lim_{x\to p}(fg)(x) = AB$;

(c) $\lim_{x\to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$, if $B \neq 0$.

Remark: If $\bar{f}, \bar{g} : E \to \mathbb{R}^k$, then (a) remains as it is, and (b) becomes $\lim_{x\to p}(\bar{f}\cdot\bar{g})(x) = A\cdot B$.

CONTINUOUS FUNCTIONS

Let X & Y be metric spaces, $E \subset X$, $p \in E$ & $f : E \to Y$. Then f is said to be continuous at p if for every $\varepsilon > 0$, $\exists\, \delta > 0 \ni d_Y(f(x), f(p)) < \varepsilon$ $\forall\, x \in E \ni d_X(x, p) < \delta$. In other words, $\lim_{x\to p} f(x) = f(p)$.

- If f is continuous at every point of E, f is said to be continuous on E.

Note: $p \in E$ in order for f to be continuous at p.

- If p is an isolated point of E, every fn. f whose domain is E is continuous at p.