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MA 509 – REAL ANALYSIS – LECTURE 31

Thm. 4.4 Consider the above defn. of f being continuous at p. Assume also that p is a limit point of E. Then f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$.

Thm. 4.5 Suppose X, Y and Z are metric spaces, $E \subset X$, f maps E into Y, g maps the range of f, $f(E)$, into Z, and $h: E \to Z$ is defined by

$$h(x) = g(f(x)) \quad (x \in E).$$

If f is continuous at a point $p \in E$ & g is continuous at $f(p)$, then h is continuous at p

Proof:— Let $\varepsilon > 0$ be given. Since g is cont. at $f(p)$, $\exists\, \eta > 0 \ni d_Z(g(y), g(f(p))) < \varepsilon$ if

$$d_Y(y, f(p)) < \eta \text{ \& } y \in f(E).$$

Since f is cont. at p, $\exists\, \delta > 0 \ni$

$$d_Y(f(x), f(p)) < \eta \text{ if } d_X(x, p) < \delta \text{ and } x \in E.$$

$\Rightarrow d_Z(h(x), h(p)) < \varepsilon$ if $d_X(x, p) < \delta$ & $x \in E$.

$\Rightarrow h$ is continuous at p. ☒

Thm. 4.6 A mapping f of a metric space X into a metric space Y is continuous on X iff $f^{-1}(V)$ is open for every open set V in Y.

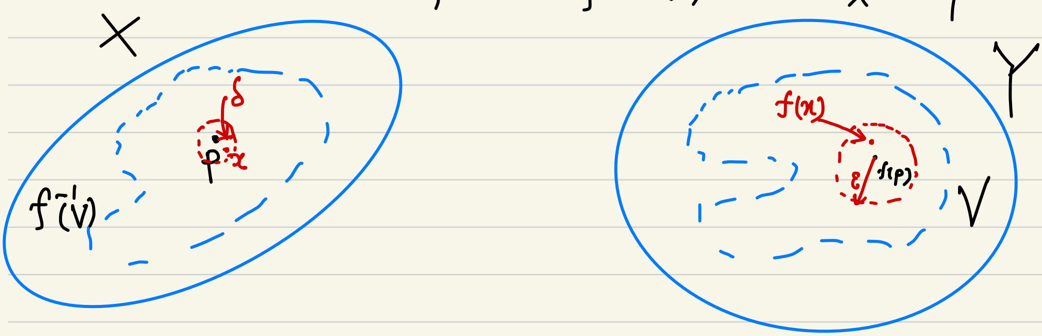
Proof: " $\Rightarrow$ " Suppose f is continuous on X. Let V be an open set in Y.

Claim: $f^{-1}(V)$ is open in X, i.e., every point of $f^{-1}(V)$ is an interior pt. of $f^{-1}(V)$.

Suppose $p \in X$ and $f(p) \in V$ (i.e., $p \in f^{-1}(V)$).
V open $\Rightarrow \exists\, \varepsilon > 0 \ni y \in V$ if $d_Y(f(p), y) < \varepsilon$.

But f is continuous at p, hence $\exists\, \delta > 0 \ni$
$d_Y(f(x), f(p)) < \varepsilon$ if $d_X(x, p) < \delta$.

i.e., $f(x) \in V$ if $d_X(x, p) < \delta$.
In other words, $x \in f^{-1}(V)$ if $d_X(x, p) < \delta$.



Thus for any $p \in f^{-1}(V)$, $\exists\, N_\delta(p) \ni$
$N_\delta(p) \subset f^{-1}(V)$.

$\Rightarrow p$ is an interior pt. of $f^{-1}(V)$.
$\Rightarrow f^{-1}(V)$ is open in X.

$\boxtimes$

Cor. 4.7 A mapping f of a metric space X into a metric space Y is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y.

Remark: For every $E \subset Y$, and f continuous on X, $f^{-1}(E^c) = (f^{-1}(E))^c$ since a set is closed iff its complement is open.

Thm. 4.8 Let f and g be complex continuous functions on a metric space X. Then $f+g$, fg and f/g are continuous on X.

(of course, while considering f/g, it is assumed that $g(x) \neq 0 \ \forall \, x \in X$.)

Thm. 4.9 (a) Let $f_1, \dots, f_k$, be real functions on a metric space X, and let $\bar{f}$ be the mapping of X into $\mathbb{R}^k$ defined by

$$\bar{f}(x) = (f_1(x), f_2(x), \dots, f_k(x)) \qquad (x \in X);$$

then $\bar{f}$ is continuous iff each of $f_1, f_2, \dots, f_k$ is continuous.

(b) If $\bar{f}$ and $\bar{g}$ are continuous mappings of X into $\mathbb{R}^k$, then $\bar{f}+\bar{g}$ and $\bar{f}\cdot\bar{g}$ are continuous on X.

Proof: (a) Note that $|f_j(x) - f_j(y)| \leq |\bar{f}(x) - \bar{f}(y)|$

$$= \left(\sum_{i=1}^{k} |f_i(x) - f_i(y)|^2 \right)^{1/2}.$$

(b) follows from (a) & the previous thm.

Examples

(1) Let $x_1, x_2, \ldots, x_k$ be the coordinates of $\bar{x} \in \mathbb{R}^k$.
Then $\phi_i : \mathbb{R}^k \to \mathbb{R}$ defined by

$$\phi_i(\bar{x}) = x_i$$

are continuous fns. on $\mathbb{R}^k$ since
$|\phi_i(\bar{x}) - \phi_i(\bar{y})| \leq |\bar{x} - \bar{y}|$ shows we can take $\delta = \varepsilon$.
These functions are called <u>coordinate fnc.</u>

(2) All polynomials, rational functions (where the denominator is non-zero on the domain) are continuous functions.

(3) $f : \mathbb{R}^k \to \mathbb{R}$ defined by $f(\bar{x}) = |\bar{x}|$ is a continuous real function on $\mathbb{R}^k$.

$$||\bar{x}| - |\bar{y}|| \leq |\bar{x} - \bar{y}| <$$

CONTINUITY AND COMPACTNESS

<u>Defn.</u> A mapping $\bar{f} : E \to \mathbb{R}^k$ is said to be <u>bounded</u> if $\exists\, M \in \mathbb{R} \ni |\bar{f}(x)| \leq M \ \forall x \in E$.

<u>Thm. 4.10</u> Suppose f is a continuous mapping of a compact metric space X in m.s. Y. Then $f(X)$ is compact.

Proof: Let $\{V_\alpha\}$ be an open cover of $f(X)$.
Since f is continuous, each $f^{-1}(V_\alpha)$ is open.

$\{f^{-1}(V_\alpha)\}$ is an open cover of X & X is cpt.

(abbreviation for compact!!)

$\Rightarrow \exists\ \alpha_1, \alpha_2, \ldots, \alpha_n \ni$

$$X \subset f^{-1}(V_{\alpha_1}) \cup f^{-1}(V_{\alpha_2}) \cup \ldots f^{-1}(V_{\alpha_n})$$

But $f(f^{-1}(E)) \subset E$ for every $E \subset Y$.

$\Rightarrow f(X) \subset V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}$.

$\Rightarrow f(X)$ is compact.

$\boxtimes$

$f(x) \in f(X)$

$f(x) \in V_\alpha$ for some α.

$x \in f^{-1}(V_\alpha)$