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MA 509 - REAL ANALYSIS - LECTURE 33

Thm. 4.15 Let E be a non-compact set in $\mathbb{R}$.
Then:
(a) $\exists$ a cont. fn. on E which is not bounded.
(b) $\exists$ a cont. and bdd. fn. on E which has no maximum.

In addition to the above hypotheses, let E be bounded. Then,
(c) $\exists$ a cont. fn. on E which is not u.c.

CASE 1 : E is bounded

Proof: Since E is non-cpt, E is infinite. Suppose first that E is bdd. Since E is not closed (why?), $\exists$ a limit pt. x_0 of E which is not in E. Consider

$$f(x) = \frac{1}{x - x_0} \quad (x \in E). \qquad \text{---} \quad (*)$$

- f is cont. on E but unbdd.

- To see that $\exists$ a continuous and bounded fn. on E which does not attain its maximum, consider

$$g(x) = \frac{1}{1 + (x - x_0)^2}, \quad x \in E.$$

Then g is cont. on E. It is also bdd as $0 < g(x) < 1$.

Also, $\sup_{x \in E} (g(x)) = 1$,

But $g(x) < 1 \ \forall x \in E$.
Hence g has no maximum on E.

- The fact that the function defined in (*) is not uniformly continuous on E is a problem in the upcoming tutorial on coming Monday (Tut. 9).

CASE 2 E is unbounded

- $f(x) = x$ on, say, $(0, \infty)$ is continuous but unbounded.

- $$h(x) = \frac{x^2}{1+x^2} \quad (x \in E)$$

is a continuous and bounded function which does not attain its maximum since $\sup_{x \in E} h(x) = 1$, and $h(x) < 1 \ \forall x \in E$.

- However (C) is not always true if E is unbounded. For example, let $E = \mathbb{Z}$. Then every fn. defined on E is uniformly continuous on E. To do so, just take $\delta < 1$ in the definition of uniform continuity.

Example Let $X = [0, 2\pi) \subset \mathbb{R}$. Let $f : X \to Y$, where Y is the unit circle, be defined by

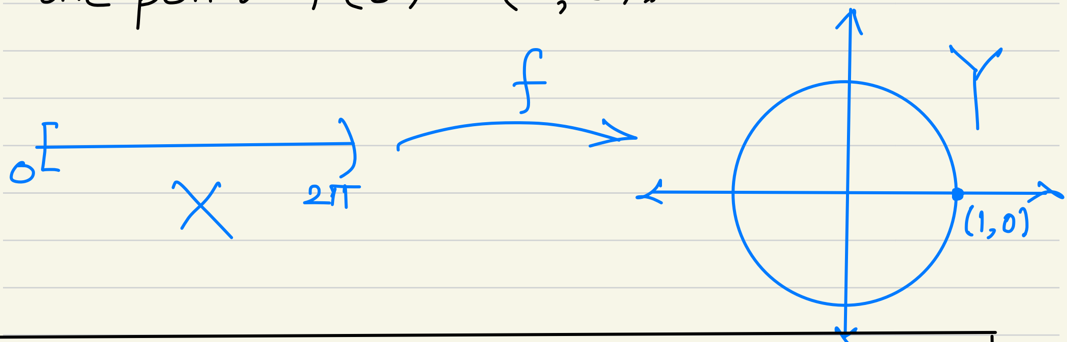
$$f(t) = (\cos t, \sin t) \quad (0 \le t < 2\pi)$$

Then f is continuous 1-1 mapping of X onto Y. [f cont. at p means given $\varepsilon > 0$, $\exists \delta > 0 \ni$ whenever $d_X(x, p) < \delta$, we have $d_Y(f(x), f(p)) < \varepsilon$

f not cont. at p means $\exists \varepsilon > 0 \ni \forall \delta > 0, \exists x \ni$

$d_X(x,p) < \delta$ but $d_Y(f(x), f(p)) \geq \varepsilon$.

But $f^{-1}: Y \to X$ is not continuous at the point $f(0) = (1, 0)$.



CONTINUITY AND CONNECTEDNESS

Thm. 4.16 If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is a connected subset of Y.

Proof: (by contradiction)

Suppose $f(E)$ is not connected. Then $f(E) = A \cup B$, where A and B are non-empty separated subsets of Y.

Let $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$.

Then $E = G \cup H$, where $G \neq \phi$ and $H \neq \phi$.

Now $A \subset \bar{A}$ implies $f^{-1}(A) \subset f^{-1}(\bar{A})$ so that

$G = E \cap f^{-1}(A) \subset E \cap f^{-1}(\bar{A})$.

$\Rightarrow$ Since f is continuous on X and $\bar{A}$ is closed in Y, $f^{-1}(\bar{A})$ is closed in X.

However, $\bar{G}$ is the smallest closed set in X containing G.

Thus $\overline{G} \subset f^{-1}(\overline{A})$.
$\Rightarrow f(\overline{G}) \subset \overline{A}$.

Now $f(H) \subset B$ and $\overline{A} \cap B = \phi$. This must mean $\overline{G} \cap H = \phi$, for if $x \in \overline{G} \cap H$, then $f(x) \in f(\overline{G}) \subset \overline{A}$, and $f(x) \in f(H) \subset B$.
$\Rightarrow f(x) \in \overline{A} \cap B = \phi$ $\rightarrow\!\leftarrow$.

This shows $\overline{G} \cap H = \phi$. Similarly, $G \cap \overline{H} = \phi$. This is not possible if E is connected.

$\blacksquare$

Thm. 4.17 INTERMEDIATE VALUE THEOREM

Let f be a continuous real function on the interval $[a,b]$. If $f(a) < f(b)$ and if c is a number s.t. $f(a) < c < f(b)$, then there exists a point $x \in (a,b) \ni f(x) = c$.

Proof: We know by a prev. thm. that intervals in $\mathbb{R}$ are connected. Thus $[a, b]$ is connected subset of $\mathbb{R}$.

But f is a continuous real function on $[a, b]$.
$\Longrightarrow f([a,b])$ is a connected subset of $\mathbb{R}$ (by Thm. 4.16).

$\Rightarrow$ if $c \in \mathbb{R} \ni f(a) < c < f(b)$, then $c \in f([a,b])$, i.e. $\exists\, x \in (a,b) \ni f(x) = c$.

$\blacksquare$

Remark: The converse of the intermediate value theorem is not true.