10/11/2020

MA 509 - REAL ANALYSIS - LECTURE 34

DISCONTINUITIES

If x is a point in the domain of defn. of a function f at which f is not continuous, we say f is <u>discontinuous</u> at x, or that f has a <u>discontinuity</u> at x.

LEFT - AND RIGHT- HAND LIMITS

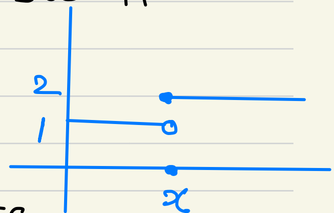
First of all, let $f(x+) := \lim_{y \to x^+} f(y)$ (right-hand limit) and $f(x-) := \lim_{y \to x^-} f(y)$ (left-hand limit).

This means the following:

- $f(x+) = q$ if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x, b) s.t. $t_n \to x$.
- Similarly, $f(x-) = q$ if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (a, x) s.t. $t_n \to x$.

Remark: Note that $\lim_{t \to x} f(t)$ exists iff

$$f(x-) = f(x+) = \lim_{t \to x} f(t).$$



TWO TYPES OF DISCONTINUITIES:

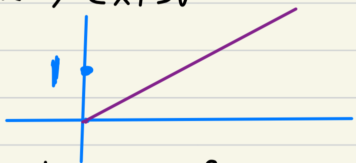
Let f be defined on (a, b). If f is discontinuous at a point x, and if $f(x+)$ and $f(x-)$ exist, then

f is said to have a discontinuity of the first kind, or a simple discontinuity at x.
Otherwise, the discontinuity is said to be of the second kind.

Examples: ① $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$

Then f has a discontinuity of the second kind since neither $f(x+)$ nor $f(x-)$ exist.

② $f(x) = \begin{cases} x & (x \in \mathbb{Q}) \\ 0 & (x \in \mathbb{R}\setminus\mathbb{Q}). \end{cases}$



Then $\lim_{x \to 0} f(x) = 0 = f(0)$. Hence f is continuous at $x=0$ and has a discontinuity of the 2^{nd} kind at every other point.

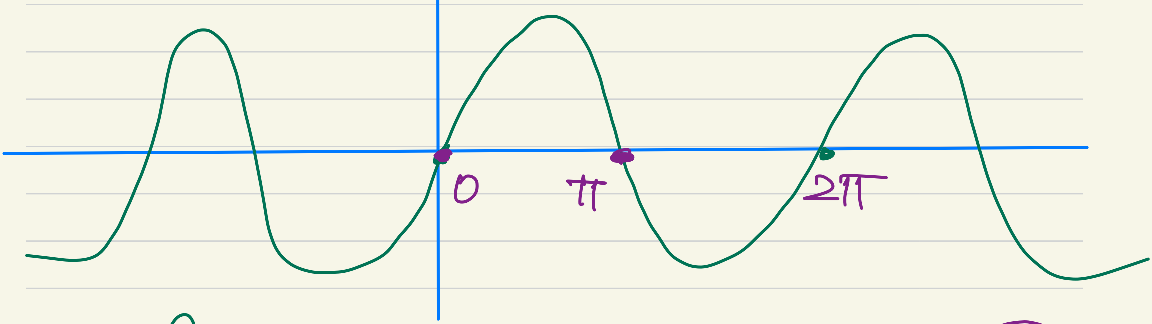
③ Let $f(x) = \begin{cases} x+2, & -3 < x < -2 \\ -x-2, & -2 \le x < 0. \\ x+2; & 0 \le x < 1 \end{cases}$

Here, $f(0+) = 2 \neq -2 = f(0-)$. Hence f has a simple discontinuity at $x=0$. Other than this, f is continuous at every point of $(-3, 1)$

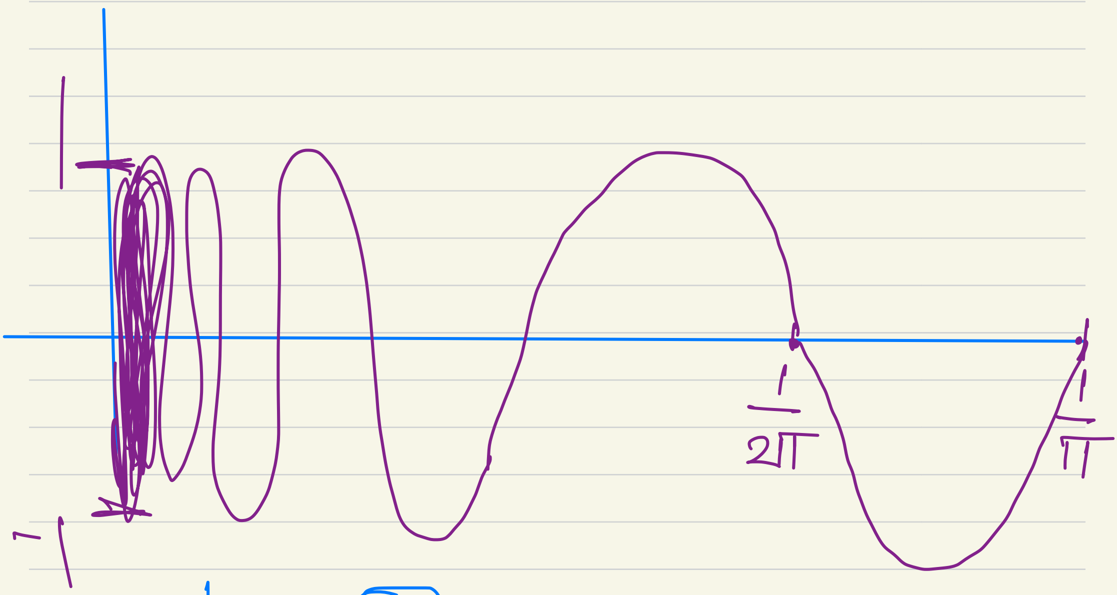
$$f(-2) = 0 = f(-2-)$$
$$0 = f(-2+)$$

④ $f(x) = \begin{cases} \sin(1/x), & x \neq 0 \\ 0 & x = 0. \end{cases}$

$g(x) = \sin x$



$f(x) = \sin\left(\frac{1}{x}\right) = 0$ for $x = \frac{1}{n\pi}$, $\forall n \in \mathbb{N}$.



$$\lim_{n\to\infty} \frac{\text{si}\cancel{n}\,(x)}{\cancel{n}} = 6$$

Both $f(0+)$ and $f(0-)$ does not exist.
Hence f has a discontinuity of the 2nd kind at $x=0$.

Other than that, f is continuous at every other point in $\mathbb{R}$.

MONOTONIC FUNCTIONS

Defn. Let f be real on (a,b). Then f is said to be monotonically increasing on (a,b) if $a<x<y<b$ implies $f(x) \le f(y)$, and monotonically decreasing if $f(x) \ge f(y)$.

Thm. 4.18 Let f be monotonically increasing on (a,b). Then $f(x+)$ and $f(x-)$ exist at every point x of (a,b). More precisely,

$$\sup_{a<t<x} f(t) = f(x-) \le f(x) \le f(x+) = \inf_{x<t<b} f(t) \quad (*)$$

Furthermore, if $a<x<y<b$, then
$$f(x+) \le f(y-).$$

- Analogous results evidently hold for monotonically decreasing functions.

Proof: Since f is monotonic, if $a<t<x$, then the set of numbers $f(t)$ are bounded above by $f(x)$.

Hence it has the least upper bound, say A.

Then $A \le f(x)$.

Claim: $A = f(x-)$.

Given $\varepsilon > 0$, since A is the least upperbound of $\{f(t) : a < t < x\}$, $\exists\ \delta > 0 \ni a < x - \delta < x$ and

$$A - \varepsilon < f(x-\delta) < A. \qquad \text{—— (1)}$$

But since f is monotonic,

$$f(x-\delta) \le f(t) \le A \qquad (x - \delta < t < x) \qquad \text{—— (2)}$$

From (1) & (2),

$$A - \varepsilon < f(t) < A + \varepsilon \qquad (x - \delta < t < x)$$

$\Rightarrow \quad |f(t) - A| < \varepsilon \quad$ for $x - \delta < t < x$.

$\Rightarrow \quad A = f(x-)$.

Similarly, one can show that

$$f(x+) = \inf_{x<t<b}(f(t)).$$

Now if $a < x < y < b$, from (*),

$$f(x+) = \inf_{x<t<b}(f(t)) = \inf_{x<t<y} f(t),$$

by applying (*) to (a, y) instead of (a, b)

Similarly, $f(y-) = \sup_{a<t<y}(f(t)) = \sup_{x<t<y}(f(t))$.

Since $\inf_{x<t<y}(f(t)) \le \sup_{x<t<y} f(t))$,

we have $f(x+) \le f(y-)$.

Cor. Monotonic functions have no discontinuities of the second kind.

Thm. 4.19 Let f be monotonic on (a,b). Then the set of points (a,b) at which f is discontinuous is at most countable.

Proof: W.l.g., f is increasing. Let E be the set of points at which f is discontinuous.

We associate to every $x \in E$, a rational number $r(x)$ $\ni$

$$f(x-) < r(x) < f(x+)$$

($\because f$ is monotonic, $f(x-)$ & $f(x+)$ both exist & since f is discontinuous, $f(x-) < f(x+)$.)

Then $x_1 < x_2$ implies $f(x_1+) \leq f(x_2-)$

$\Rightarrow r(x_1) \neq r(x_2)$ whenever $x_1 \neq x_2$.

Thus $\exists$ a 1-1 correspondence between set of discontinuities of f and the set of rational numbers. $\blacksquare$